① These are just examples. your solutions will vary.

I will show that "A is invertible" $\Leftrightarrow$ "A is row equivalent to the $n \times n$ identity" $\Leftrightarrow$ "A has n pivots" $\Leftrightarrow$ "A^T is invertible".

a. To show that "A is invertible" $\Leftrightarrow$ "A^T is invertible" we can appeal to Theorem 6 in the textbook. (part c)

If A is invertible, then $A^{-1}A = I$ and $AA^{-1} = I$. If we transpose both of these statements we get $(A^{-1}A)^T = I^T \Rightarrow$ $A^T(A^{-1})^T = A^T(A^T)^{-1} = I$ and $(AA^{-1})^T = I^T \Rightarrow (A^{-1})^T A^T =$ $(A^T)^{-1}A^T = I$. Since this equation satisfies the inverse property for A^T, A^T is also invertible. By a similar calculation we find the reverse is also true. If $A^T(A^T)^{-1} = I$ and $(A^T)^{-1}A^T = I$ then by transposing both sides of each statement we obtain

$$\left(A^T(A^T)^{-1}\right)^T = I^T \Rightarrow \left[(A^T)^{-1}\right]^T (A^T)^T = \left[(A^T)^T\right]^{-1} A = A^{-1}A = I \text{ and also}$$

$$\left((A^T)^{-1}A^T\right)^T = I^T \Rightarrow (A^T)^T\left[(A^T)^{-1}\right]^T = A\left[(A^T)^T\right]^{-1} = AA^{-1} = I. \ //$$

b. To show that "A is invertible" $\Leftrightarrow$ "A is row equivalent to the $n \times n$ identity" we can appeal to Theorem 7 in the textbook (this is the theorem that justifies our inverse finding algorithm).

Suppose that A is invertible, then by Theorem 7, A is invertible if and only iff A is row equivalent to I_n, which is shorthand for the $n \times n$ Identity. "If and only if" statements are bidirectional.

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c. If we wish to show that "A is row equivalent to the nxn identity" is equivalent to "A has n pivots" we only need to think about what A must look like in echelon form and compare to the identity. an nxn identity is a matrix w/ 1's on the diagonal and 0's everywhere else

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & \vdots \\ \vdots & \vdots & 1 & 0 \\ 0 & 0 & \cdots & 1 \end{bmatrix}$$

i.e. entry $(I_n)_{ii} = 1$ and $(I_n)_{ij}, i \neq j = 0$.

if we count the number of pivots, there are n of them. Similarly, if we take a nxn matrix A and reduce it to echelon form and find n pivots, we can continue reducing further to leave only 1's in the pivots and 0's elsewhere (from 1.2) for each of the pivot columns Since there are n pivots and n columns, that means each column has a single one and all zeros. Since reduced echelon form is unique, the locations of those pivots are fixed at a_{ii}, in other words, the reduced echelon form is identical to the identity. //

2. Suppose A, B are nxn, Then AB is defined and is also nxn. if AB is invertible then $(AB)^{-1}(AB) = I$ and $(AB)(AB)^{-1} = I$

But this implies that $(B^{-1}A^{-1})(AB) = I$ by associativity

$B^{-1}(A^{-1}A)B = I \Rightarrow B^{-1}IB = I \Rightarrow B^{-1}B = I$, Similarly

$AB(B^{-1}A^{-1}) = I \Rightarrow ABB^{-1}A^{-1} = I$ multiply on the right side by $A \Rightarrow ABB^{-1}(A^{-1}A) = IA \Rightarrow ABB^{-1}I = A \Rightarrow ABB^{-1} = A$

and multiply on the left by $A^{-1} \Rightarrow (A^{-1}A)BB^{-1} = (A^{-1})A \Rightarrow$

$IBB^{-1} = I \Rightarrow BB^{-1}I$. Since we have both $B^{-1}B = I$ and $BB^{-1} = I$, B is invertible. //

3. Suppose that T is a linear transformation that maps $R^n \to R^n$, and T is onto. T has can represented as a matrix A so that $\vec{x} \in \mathbb{R}^n \to A\vec{x} \in \mathbb{R}^n$. If A is onto, then it has n pivots and since A must be $n \times n$, this implies by the invertible matrix theorem that A is invertible. If we define $T^{-1}(\vec{b})$: $\vec{b} \in \mathbb{R}^n \to A^{-1}\vec{b} = \vec{x} \in \mathbb{R}^n$. But if A is invertible so is A^{-1}, and by the invertible matrix theorem applied to A^{-1}, it must be one-to-one. //

4. let $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$ and $C = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$

and let $B = \begin{bmatrix} a_{21} & a_{22} & a_{23} \\ a_{11} & a_{12} & a_{13} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$ ($R_1 \Leftrightarrow R_2$ in A.)

$\det A = (a_{11})(a_{22}a_{33} - a_{32}a_{23}) - (a_{12})(a_{21}a_{33} - a_{31}a_{22}) + (a_{13})(a_{21}a_{32} - a_{31}a_{22})$

$\det B = (a_{21})(a_{12}a_{33} - a_{32}a_{13}) - (a_{22})(a_{11}a_{33} - a_{31}a_{13}) + (a_{23})(a_{11}a_{32} - a_{31}a_{12})$

if you check very carefully every term in $\det B$ has the opposite sign of $\det A$, thus $\det B = -\det A$ or $\det A = -\det B$.

without loss of generality, it's easy to show that the results are the same for exchanging $R_1 \Leftrightarrow R_3$, or $R_2 \Leftrightarrow R_3$. These are the only possible row exchanges. //

b. $\det A$ is above, so compare to $\det A^T = \det \begin{bmatrix} a_{11} & a_{21} & a_{31} \\ a_{12} & a_{22} & a_{32} \\ a_{13} & a_{23} & a_{33} \end{bmatrix} =$

$(a_{11})(a_{22}a_{33} - a_{23}a_{32}) - (a_{21})(a_{12}a_{33} - a_{13}a_{32}) + (a_{31})(a_{12}a_{23} - a_{13}a_{22})$

Carefully checking all 6 terms, they are identical.

4c. Suppose $A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$ and $C = \begin{bmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{bmatrix}$ for this part only

$$AC = \begin{bmatrix} a_{11}c_{11} + a_{12}c_{21} & a_{11}c_{12} + a_{12}c_{22} \\ a_{21}c_{11} + a_{22}c_{21} & a_{21}c_{12} + a_{22}c_{22} \end{bmatrix}$$

$$\det AC = (a_{11}c_{11} + a_{12}c_{21})(a_{21}c_{12} + a_{22}c_{22}) - (a_{21}c_{11} + a_{22}c_{21})(a_{11}c_{12} + a_{12}c_{22})$$

$$= \cancel{a_{11}c_{11}a_{21}c_{12}} + a_{11}c_{11}a_{22}c_{22} + a_{12}c_{21}a_{21}c_{12} + \cancel{a_{12}c_{21}a_{22}c_{22}}$$

$$- \cancel{a_{21}c_{11}a_{11}c_{12}} - a_{21}c_{11}a_{12}c_{22} - a_{22}c_{21}a_{11}c_{12} - \cancel{a_{22}c_{21}a_{12}c_{22}}$$

$$= a_{11}c_{11}a_{22}c_{22} + a_{12}c_{21}a_{21}c_{12} - a_{21}c_{11}a_{12}c_{22} - a_{22}c_{21}a_{11}c_{12}$$

$\det A = (a_{11}a_{22} - a_{21}a_{12})$ $\quad \det C = (c_{11}c_{22} - c_{21}c_{12})$

$(\det A)(\det C) = a_{11}c_{11}a_{22}c_{22} - a_{11}a_{22}c_{21}c_{12} - a_{21}a_{12}c_{11}c_{22} + a_{21}a_{12}c_{21}c_{12}$

which are equal. //

d. using part c above $\det A^2 = \det(AA) = (\det A)(\det A) = (\det A)^2$; similarly for $\det A^3 = \det(A^2 A) = (\det A^2)(\det A) = (\det A)^2(\det A) = (\det A)^3$. by mathematical induction, assume $\det A^n = (\det A)^n$, we can show this works for $\det A^{n+1}$! since $\det(A^{n+1}) = \det(A^n \cdot A) = (\det A^n)\det A = (\det A)^n(\det A) = (\det A)^{n+1}$ //

e. $\det(rA) = \det \begin{bmatrix} ra_{11} & ra_{12} & ra_{13} \\ ra_{21} & ra_{22} & ra_{23} \\ ra_{31} & ra_{32} & ra_{33} \end{bmatrix} =$

$$ra_{11}(ra_{22}ra_{33} - ra_{32}ra_{23}) - ra_{12}(ra_{21}ra_{33} - ra_{31}ra_{23}) + ra_{31}(ra_{22}ra_{33} - ra_{32}ra_{23})$$

$$= r^3\left[a_{11}(a_{22}a_{33} - a_{32}a_{23}) - a_{12}(a_{21}a_{33} - a_{31}a_{23}) + a_{31}(a_{22}a_{33} - a_{32}a_{23})\right] =$$

$r^3 \det A$. (this generalize to $n \times n$ matrices $\det(rA) = r^n A$. //

5. all polynomials of at most degree 3 $p(t) = a_0 + a_1 t + a_2 t^2 + a_3 t^3$. (5)

$\Longleftrightarrow \begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$. entries above $a_3 = 0$.

a. contains the zero vector if $a_0 = a_1 = a_2 = a_3 = 0$

b. $\begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \\ 0 \\ \vdots \\ 0 \end{bmatrix} + \begin{bmatrix} b_0 \\ b_1 \\ b_2 \\ b_3 \\ 0 \\ \vdots \\ 0 \end{bmatrix} = \begin{bmatrix} a_0 + b_0 \\ a_1 + b_1 \\ a_2 + b_2 \\ a_3 + b_3 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \in \mathbb{P}_n$ since

$(a_0 + b_0) + (a_1 + b_1)t + (a_2 + b_3)t^2 + (a_3 + b_3)t^3$

is still a degree-3 poly.

c. $k \begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \\ 0 \\ \vdots \\ 0 \end{bmatrix} = \begin{bmatrix} ka_0 \\ ka_1 \\ ka_2 \\ ka_3 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \in \mathbb{P}_n$ since $ka_0 + ka_1 t + ka_2 t^2 + ka_3 t^3$

is still a degree-3 polynomial //