Math 2568 Vector Spaces & Subspaces Key ①

1. $V = \left\{ \begin{bmatrix} a \\ b \\ c \end{bmatrix}, a = b + c \right\}$

i) $\begin{bmatrix} b+c \\ b \\ c \end{bmatrix}$ let $b, c = 0 \Rightarrow \begin{bmatrix} 0+0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$ zero is in the set

ii) $\begin{bmatrix} b+c \\ b \\ c \end{bmatrix} + \begin{bmatrix} a+d \\ a \\ d \end{bmatrix} = \begin{bmatrix} b+c+a+d \\ b+a \\ c+d \end{bmatrix}$ does $b+c+a+d = (b+a) + (c+d)$? yes.

iii) $k \begin{bmatrix} b+c \\ b \\ c \end{bmatrix} = \begin{bmatrix} k(b+c) \\ kb \\ kc \end{bmatrix}$ does $k(b+c) = kb + kc$? yes.

V is a vector space/subspace

2. ii) $\vec{b}_1 = A\vec{x} = \begin{bmatrix} 1 & 4 & 1 & 0 \\ 0 & 2 & 1 & -1 \\ 0 & 3 & -2 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} x_1 + 4x_2 + x_3 \\ 2x_1 + x_3 - x_4 \\ 3x_2 - 2x_3 \end{bmatrix}$

$\vec{b}_2 = A\vec{y} = \begin{bmatrix} 1 & 4 & 1 & 0 \\ 0 & 2 & 1 & -1 \\ 0 & 3 & -2 & 0 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{bmatrix} = \begin{bmatrix} y_1 + 4y_2 + y_3 \\ 2y_1 + y_3 - y_4 \\ 3y_2 - 2y_3 \end{bmatrix}$

$\vec{b}_1 + \vec{b}_2 = \begin{bmatrix} x_1 + 4x_2 + x_3 + y_1 + 4y_2 + y_3 \\ 2x_1 + x_3 - x_4 + 2y_1 + y_3 - y_4 \\ 3x_2 - 2x_3 + 3y_2 - 2y_3 \end{bmatrix}$

$A(\vec{x} + \vec{y}) = \begin{bmatrix} 1 & 4 & 1 & 0 \\ 0 & 2 & 1 & -1 \\ 0 & 3 & -2 & 0 \end{bmatrix} \begin{bmatrix} x_1 + y_1 \\ x_2 + y_2 \\ x_3 + y_3 \\ x_4 + y_4 \end{bmatrix} = \begin{bmatrix} x_1 + y_1 + 4x_2 + 4y_2 + x_3 + y_3 \\ 2x_1 + 2y_1 + x_3 + y_3 - x_4 - y_4 \\ 3x_2 + 3y_2 - 2x_3 - 2y_3 \end{bmatrix}$

Shows that $b_1 + b_2$ in X.

iii) $k\vec{b}_1 = k(A\vec{x}) = A(k\vec{x})$

$k \begin{bmatrix} x_1 + 4x_2 + x_3 \\ 2x_1 + x_3 - x_4 \\ 3x_2 - 2x_3 \end{bmatrix} = \begin{bmatrix} kx_1 + 4x_2 + kx_3 \\ 2kx_1 + kx_3 - kx_4 \\ 3kx_2 - 2kx_3 \end{bmatrix}$ ✓ $k\vec{b}$ is in X

i) $A\vec{0} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$ zero vector in X.

X is a subspace

②

3. $T(\vec{x}) = \begin{bmatrix} 3x_1 - 2x_4 \\ 6x_2 \\ -1 \\ x_3 - x_4 \end{bmatrix}$ $\vec{x} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$

i) $T(\vec{0}) = \begin{bmatrix} 0 \\ 0 \\ -1 \\ 0 \end{bmatrix}$ $\vec{0}$ is not in the set $\Rightarrow$ Q is not a subspace

4. $Y \{ \begin{bmatrix} x \\ y \end{bmatrix}, xy \geq 0 \}$

i) $x, y = 0$ $\begin{bmatrix} 0 \\ 0 \end{bmatrix}$ $\vec{0}$ is in the space

ii) consider vectors $\vec{u} = \begin{bmatrix} 7 \\ 1 \end{bmatrix}$ and $\vec{v} = \begin{bmatrix} -1 \\ -4 \end{bmatrix}$ both vectors are in Y

but $\vec{u} + \vec{v} = \begin{bmatrix} 7 \\ 1 \end{bmatrix} + \begin{bmatrix} -1 \\ -4 \end{bmatrix} = \begin{bmatrix} 6 \\ -3 \end{bmatrix}$ which is not in Y since $6(-3) \not\geq 0$

therefore Y is not closed under addition and so it is not a subspace

5. $Z = \{ \begin{bmatrix} x \\ y \end{bmatrix}, x^2 + y^2 \geq 1 \}$

i) $\begin{bmatrix} 0 \\ 0 \end{bmatrix}$ $0^2 + 0^2 \not\geq 1$ therefore The $\vec{0}$ is not in the set

therefore Z is not a subspace. (also fails scalar multiplication for scalars $|k| < 1$)

6. $S = \{ \begin{bmatrix} a & b & c \\ -b & a & d \end{bmatrix}, a + b + c = d \}$

i) let $a, b, c, d = 0$ $\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ is the zero vector $\ni$ $0 + 0 + 0 = 0$

ii) $\begin{bmatrix} a & b & c \\ -b & a & d \end{bmatrix} + \begin{bmatrix} e & f & g \\ -f & e & h \end{bmatrix} = \begin{bmatrix} a+e & b+f & c+g \\ -b-f & a+e & d+h \end{bmatrix}$

$(a+e) + (b+f) + (c+g) = (d+h)$ true since

$(a+b+c) + (e+f+g) = d + h$ S closed under addition

6 contd

iii) $k\begin{bmatrix} a & b & c \\ -b & a & d \end{bmatrix} = \begin{bmatrix} ka & kb & kc \\ -kb & ka & kd \end{bmatrix}$ $ka + kb + kc = k(a+b+c) = kd$

closed under scalar multiplication

S is a subspace

7. $T = \left\{ \begin{bmatrix} a & 2 \\ 0 & b \end{bmatrix},\ a, b \text{ real} \right\}$

i) the zero vector is not in the set $a, b = 0 \Rightarrow \begin{bmatrix} 0 & 2 \\ 0 & 0 \end{bmatrix} \neq \vec{0}$

T is not a subspace.

8. $U = \left\{ \begin{bmatrix} a \\ b^3 \end{bmatrix},\ a, b \text{ real} \right\}$

i) $a, b = 0$ $\begin{bmatrix} 0 \\ 0 \end{bmatrix}$ $\vec{0}$ vector is in the set

ii) $\begin{bmatrix} a \\ b^3 \end{bmatrix} + \begin{bmatrix} c \\ d^3 \end{bmatrix} = \begin{bmatrix} a+c \\ b^3+d^3 \end{bmatrix}$ $a+c$ is real.
b^3+d^3 is real and can be represented as the cube of real numbers.
closed under addition

iii) $k\begin{bmatrix} a \\ b^3 \end{bmatrix} = \begin{bmatrix} ka \\ kb^3 \end{bmatrix}$ ka is real. kb^3 is real & can be represented as the cube of real numbers
closed under scalar multiplication

U is a subspace

9. $B = \left\{ \begin{bmatrix} a+2 & b \\ c & d \end{bmatrix},\ a+b+c=d \right\}$

i) let $a, b, c, d = 0$ $0+0+0=0$ but $\begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix}$ is not the zero vector, nor does it helps if we let $a=-2$, since this would force other entries to be nonzero.

therefore, $\vec{0}$ is not in the set

B is not a subspace

10. $\mathbb{C} = \{a+bi,\ a,b \text{ real}\}$

i) if $a, b = 0$ $\quad 0 + 0i = 0$ which is the 0 vector in the set.

ii) $(a+bi) + (c+di) = (a+c) + (b+d)i$, $a+c$ real & $b+d$ real
So closed under addition

iii) $k(a+bi) = ka + (kb)i$ $\quad ka$ is real, kb is real
therefore closed under scalar multiplication

$\mathbb{C}$ is a vector space.

11. $D = \{p(t) = at^2\}$

i) if $a=0$, $p(t)=0$ which is the zero vector

ii) $at^2 + bt^2 = (a+b)t^2$, $a+b$ is real so closed under addition

iii $k(at^2) = (ka)t^2$ $\quad ka$ is real, so closed under scalar multiplication

D is a subspace.

12. $A = \{p(t)$ where $p(t)$ is polynomial of degree exactly 2$\}$

i) $p(t)$ of the form at^2, but if $a=0$, $p(t)=0$ is not of degree 2, so no zero in the set.

A is not a subspace

13. $G = \{p(t)$ degree less than 5, greater than 1$\}$
i.e. $p(t) = a_2t^2 + a_3t^3 + a_4t^4$

i) $p(t)=0$ is the zero vector, but it can't in the set since this polynomial is not of degree 2, 3 or 4.

G is not a subspace.

14. $J = \{ p(t)$ divisible by $(t-1) \}$

Consider $p(t) = (t-1)q(t)$ where $q(t)$ is a polynomial in $\mathbb{P}_n$.

i) if $q(t) = 0$ then $p(t) = 0$ $\vec{0}$ is in the set

ii) $p_1(t) = (t-1)q_1(t)$, $p_2(t) = (t-1)q_2(t)$

$p_1(t) + p_2(t) = (t-1)q_1(t) + (t-1)q_2(t) = (t-1)[q_1(t) + q_2(t)]$

Since $q_1(t) + q_2(t)$ is a polynomial, this is a polynomial divisible by $t-1$, so closed under addition

iii) $kp(t) = k(t-1)q(t) = (t-1)[kq(t)]$; $kq(t)$ is a polynomial so satisfies the set conditions

J is a subspace.

15. $K = \{$ the set of all functions w/ an x-intercept at $x=3 \}$

another way to think of this set is $f(3) = 0$

i) $f(x) = 0$ certainly has $f(3) = 0$ so $\vec{0}$ is in the set

ii) $f(x) + g(x)$ is in the set since $f(3) + g(3) = 0 + 0 \Rightarrow$ $(f+g)(3) = 0$ closed under addition

iii) $kf(x)$ is in the set since $kf(3) = k(0) = 0$ so it has an intercept at $x=3$ also. closed under scalar multiplication.

So K is a subspace

16. $L = \{$ Set of functions w/ y-intercept at $0 \}$

i.e. $f(0)=0$.

this problem solves just like #15

i) $f(x)=0$ is in the set since $f(0)=0$

ii) $f(x)+g(x) = (f+g)(x)$ is in the set since $f(0)+g(0) =$
$0+0 = (f+g)(0)$ closed under addition

iii) $kf(x)$ is in the set since $kf(0) = k(0) = 0$. closed under scalar multiplication.

17. $P = \{$ set of functions w/ $f(0)=2 \}$

i) $\vec{0}$ is not in the set since $f(x)=0$ does not have a y-intercept at 2.

this set fails all three tests. P is not a subspace

18. $R = \{$ Set of all convergent definite integrals $\}$

another way to put this is all integrals $\int_a^b f(x)\,dx$
for $a, b \in (-\infty, \infty)$ such that $\int_a^b f(x)\,dx = L$ and L is finite.

i) $\int_a^b f(x)\,dx$ for $f(x)=0 \Rightarrow \int_a^b f(x)\,dx = 0$ or, $a=b$
will make $L=0$ so $\vec{0}$ is in the set.

ii) $\int_a^b f(x)\,dx + \int_c^d g(x)\,dx$ is in the set since $\int_a^b f(x)\,dx = L$
and $\int_c^d g(x)\,dx = M$ then $\int_a^b f(x)\,dx + \int_c^d g(x)\,dx = L+M$
which is finite since L and M are. closed under addition

18 cont'd

iii) $k\int_a^b f(x)\,dx = kL$ since k is real its finite, so kL is too closed under scalar multiplication

19. $E = \{$ set of all even functions $\}$

i) $f(x)=0$ is an even function, so it is in the set

ii) $f(x)+g(x) = (f+g)(x) = (f+g)(-x)$ since $f(x)=f(-x)$ and $g(x)=g(-x)$ so $f(x)+g(x) = f(-x)+g(-x) = (f+g)(-x)$

So sum of even functions is even ∴ closed under addition

iii) $kf(x) = kf(-x)$ so closed under scalar multiplication

E is a subspace

20. $O = \{$ set of all odd functions $\}$

i) does not contain the $\vec{0}$ since $f(x)=0$ is not even

21. $\Delta = \{ f(x) = ax^{-n} = \frac{a}{x^n} \}$

i) if $a=0$, $f(x)=0$ is in the set

ii) consider $f(x) = \frac{1}{x}$ & $g(x) = \frac{1}{x^2}$

$f(x)+g(x) = \frac{1}{x} + \frac{1}{x^2} = \frac{x+1}{x^2}$ which is not in the set

Δ is not a subspace

22. $\Omega = \{$ set of all functions w/ asymptote at $x=-2 \}$

i) $\vec{0}$ not in the set since $f(x)=0$ has no asymptotes

Ω is not a subspace

23. $\Gamma = \{$ Set of all functions defined on $[0,1]$ $\}$ (8)

i) $f(x)=0$ is defined on $[0,1]$ so it is in the set

ii) $f(x)+g(x)$ is defined on $[0,1]$ if both $f(x), g(x)$ are closed under addition

iii) $kf(x)$ is defined on $[0,1]$ if $f(x)$ is, so closed under Scalar multiplication.

Γ is a subspace.