Math 1152, Exam #3, Spring 2014 Name KEY

Instructions: Show all work. Use exact answers unless specifically asked to round. Answers with no work cannot receive full credit. Be sure to answer all the requested elements of each question.

1. List the first five terms of the sequences below. (6 points each)

   a. $a_n = \frac{(-2)^{n+1}}{2^n+1}$

$n=0 \quad \frac{(-2)^1}{2^0+1} = \frac{-2}{2} = -1$

$n=1 \quad \frac{(-2)^2}{2+1} = \frac{4}{3}$

$n=2 \quad \frac{(-2)^3}{5} = \frac{-8}{5}$

$n=3 \quad \frac{(-2)^4}{9} = \frac{16}{9}$

$n=4 \quad \frac{(-2)^5}{16+1} = \frac{-32}{17}$

$-1, 4/3, -8/5, 16/9, -32/17, \ldots$

   b. $b_{n+1} = 2b_n^2 - n + 1, b_0 = 1$

$b_1 = 2(1)^2 - (0) + 1 = 3$

$b_2 = 2(3)^2 - 1 + 1 = 18$

$b_3 = 2(18)^2 - 2 + 1 = 647$

$b_4 = 2(647)^2 - 3 + 1 = 837{,}216$

$b_5 = 2(837216)^2 - 4 + 1 = 1.4 \times 10^{12}$

$1, 3, 18, 647, 837216, 1.4\times10^{12}, \ldots$

2. Find the limit of the sequence if it exists. If it does not, state that it diverges. (6 points each)

   a. $a_n = \frac{1}{2}\arctan(n)$

$\frac{1}{2}\lim_{n\to\infty} \arctan n = \frac{1}{2}\left(\pi/2\right) = \pi/4$

   b. $b_n = \frac{(-1)^{n+1} n}{\sqrt{n^5+3}}$

$\lim_{n\to\infty} (-1)^{n+1} \cdot \lim_{n\to\infty} \frac{n}{\underbrace{\sqrt{n^5+3}}_{\approx n^{5/2}}} = 0$

c. $c_n = \ln(\sin\frac{1}{n}) + \ln(n)$

$\lim_{n\to\infty} \ln \sin\frac{1}{n} + \underbrace{\lim_{n\to\infty} \ln n}_{-\infty + \infty \text{ indeterminant}} = \lim_{n\to\infty} \ln\left[\left(\sin\frac{1}{n}\right)\cdot n\right] =$

$\lim_{n\to\infty} \ln\left[\frac{\sin\frac{1}{n}}{\frac{1}{n}}\right] = \ln \lim_{n\to\infty}\left(\frac{\sin\frac{1}{n}}{\frac{1}{n}}\right) = \ln(1) = 0$

3. Find the first 5 terms of the partial sum and find an expression for S_n for the series $\sum_{k=1}^{\infty} \frac{2}{(2k+3)(2k+1)}$. (7 points)

$\frac{2}{5\cdot 3} + \frac{2}{7\cdot 5} + \frac{2}{9\cdot 7} + \frac{2}{11\cdot 9} + \frac{2}{13\cdot 11}$

$S_1 = \frac{2}{15}$ $S_2 = \frac{4}{21}$ $S_3 = 2/9$ $S_4 = 8/33$ $S_5 = 10/39$

$\frac{A}{2k+3} + \frac{B}{2k+1} \Rightarrow A(2k+1) + B(2k+3) = 2$

$k = 1/2$ $B(2) = 2 \Rightarrow B = 1$

$k = -3/2$ $A(-2) = 2 \Rightarrow A = -1$

$\sum_{k=1}^{n}\left(\frac{1}{2k+1} + \frac{-1}{2k+3}\right) = S_n = \frac{1}{3} - \frac{1}{2n+3}$

4. To what value (if any) does the geometric series $\sum_{k=0}^{\infty} 2^n 3^{-n}$ converge? If it fails to converge, explain why. (5 points)

$\sum_{k=0}^{\infty} 2^n 3^{-n} = \sum_{k=0}^{\infty}\left(\frac{2}{3}\right)^n$ $r = \frac{2}{3} < 1$ does converge

$S = \frac{1}{1-2/3} = \frac{3}{3-2} = 3$

5. Determine if the infinite series converge or diverge. Explain your reasoning, and which test you used to determine it. (8 points each)

a. $\sum_{k=2}^{\infty} \frac{\sqrt{k}}{ln^{10}(k)}$

$$\lim_{k \to \infty} \frac{k^{1/2}}{\ln^{10}(k)} = \lim_{k \to \infty} \frac{\frac{1}{2}k^{-1/2}}{10 \ln^{9}(k) \cdot 1/k} \dots$$

by continuing L'Hopital's k will survive longer than $\ln^{n}(k)$

$\ln^{3} k << k^{s}$

by dominance this limit $\to \infty$

diverges by nth term test

b. $\sum_{k=1}^{\infty} ke^{-k^2}$

$$\int_1^{\infty} k e^{-k^2} dk \qquad u = -k^2 \quad -\tfrac{1}{2} du = \cancel{2}k\, dk$$

$$\int -\tfrac{1}{2} e^{u} du = -\tfrac{1}{2} e^{-k^2} \Big|_1^{\infty} = -\tfrac{1}{2}(0) + \tfrac{1}{2} e^{-1}$$

converges by integral test.

c. $\sum_{k=1}^{\infty} \frac{2^k k!}{k^k}$

$$\lim_{k \to \infty} \left| \frac{2^{k+1}(k+1)!}{(k+1)^{k+1}} \cdot \frac{k^k}{2^k k!} \right| = \left| \frac{2\cancel{(k+1)}k^k}{\cancel{(k+1)}(k+1)^k} \right|$$

$$= 2 \lim_{k \to \infty} \left| \left(\frac{k}{k+1} \right)^k \right| = 2 \lim_{k \to \infty} \left(1 - \frac{1}{k}\right)^k = 2e^{-1} = \frac{2}{e} < 1$$

Converges by ratio test

d. $\sum_{k=1}^{\infty} sin^2\left(\frac{1}{k}\right)$

Compare w/ $\frac{1}{k^2}$ (converges by p-test)

$$\lim_{k \to \infty} \frac{\sin^2 \frac{1}{k}}{1/k^2} = \lim_{k \to \infty} \frac{\sin(1/k)}{1/k} \cdot \frac{\sin(1/k)}{1/k} = 1$$

so $\sum_{k=1}^{\infty} \sin^2(1/k)$ converges also

6. Determine if the alternating series converges conditionally, absolutely or diverges. Explain your reasoning. (7 points each)
   a. $\sum_{k=1}^{\infty}(-1)^k k^{1/k}$

$\lim_{k\to\infty} k^{1/k} = \lim_{k\to\infty} \sqrt[k]{k} = 1$ diverges since limit is not 0

   b. $\sum_{k=1}^{\infty}\frac{(-1)^k}{\sqrt{k}}$ $\lim_{k\to\infty} \frac{1}{\sqrt{k}} = 0$

this converges by the alternating series test, but $\sum_{k=1}^{\infty}\left|\frac{(-1)^k}{\sqrt{k}}\right| = \sum_{k=1}^{\infty}\frac{1}{\sqrt{k}}$ diverges by the p-test

$\therefore$ Converges conditionally

7. Find the minimum number of terms that would be needed to approximate each series to better than three decimal places accuracy (i.e. R<0.001). (8 points each)
   a. $\sum_{n=3}^{\infty}\frac{1}{k\ln k\, ln\,(\ln k)}$

$\int_{n+1}^{\infty}\frac{1}{x\ln x \ln(\ln x)}\,dx$ $u = \ln(\ln x)$, $du = \frac{1}{x\ln x}\,dx$

$\int \frac{1}{u}\,du = \ln u \Rightarrow \ln(\ln(\ln x))\Big|_{n+1}^{\infty} = \infty$

this series cannot be approximated since it diverges

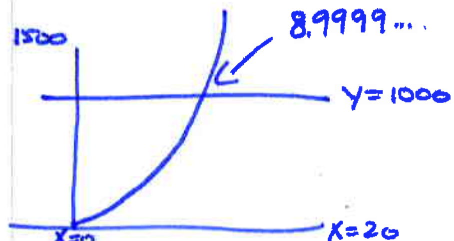
b. $\sum_{k=1}^{\infty} \frac{(-1)^k k}{k^4+1}$

$$\frac{n+1}{(n+1)^4+1} < .001$$

$$\boxed{n \geq 9}$$

$$\frac{(n+1)^4+1}{n+1} > 1000$$

graphing



8. Find the first 4 non-zero terms for the Taylor polynomial that approximates $y = \sinh(x)$. [Note: there is at least two ways to do this that we have learned. You may use either method.] (10 points)

$c=0$

n	$n!$	$f^{(n)}(x)$	$f^n(0)$	x^n	term
0	1	$\sinh x$	0	1	0
1	1	$\cosh x$	1	x	x
2	2	$\sinh x$	0	x^2	0
3	6	$\cosh x$	1	x^3	$x^3/6$
					⋮
					etc.

$$P_7(x) = x + \frac{x^3}{3!} + \frac{x^5}{5!} + \frac{x^7}{7!}$$

OR

$$\sinh x = \frac{e^x - e^{-x}}{2} = \frac{1}{2}\left[\left(1 + x + \frac{x^2}{2} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} + \frac{x^6}{6!} + \frac{x^7}{7!}\right) - \left(1 - x + \frac{x^2}{2} - \frac{x^3}{3!} + \frac{x^4}{4!} - \frac{x^5}{5!} + \frac{x^6}{6!} - \frac{x^7}{7!}\right)\right]$$

$$e^x = \sum \frac{x^n}{n!}$$

$$e^{-x} = \sum \frac{(-1)^n x^n}{n!}$$

$$= \frac{1}{2}\left[0 + 2x + 0 + \frac{2x^3}{3!} + 0 + \frac{2x^5}{5!} + 0 + \frac{2x^7}{7!}\right] =$$

$$x + \frac{x^3}{3!} + \frac{x^5}{5!} + \frac{x^7}{7!} = P_7(x)$$

9. Rewrite the expression $y = \frac{x}{(x^2+1)^2}$ as a power series. (10 points)

$$\sum_{n=0}^{\infty} ar^n = a(1-r)^{-1}$$

$$\sum_{n=1}^{\infty} anr^{n-1} = a(1-r)^{-2} \Rightarrow \sum_{k=0}^{\infty} a(k+1)r^k = a/(1-r)^2$$

$$y = \frac{x}{(1+x^2)^2} \Rightarrow \sum_{k=0}^{\infty} x \cdot (k+1)(-x^2)^k$$

$\uparrow$ $r = -x^2$

$a = x$

$$= \sum_{k=0}^{\infty} (k+1)(-1)^k x^{2k+1}$$

10. Given that the Taylor series for $e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$, use this fact to write the function e^{-x^2} as a power series and find an expression for the integral $\int e^{-x^2} dx$. (10 points)

$$e^{-x^2} = \sum_{n=0}^{\infty} \frac{(-x^2)^n}{n!} = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{n!}$$

$$\int e^{-x^2} dx = \int \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{n!} dx = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \int x^{2n} dx =$$

$$\sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \frac{x^{2n+1}}{(2n+1)} + C$$

11. What is the maximum error for the Taylor polynomial $\ln(1+x) \approx x + \frac{x^2}{2} + \frac{x^3}{3} + \frac{x^4}{4}$ on the interval $\cancel{\left[-\frac{1}{2}, 1\right]}$. (9 points) $[-1/2, 1/2]$

$$P_4(x) = x + \frac{x^2}{2} + \frac{x^3}{3} + \frac{x^4}{4} + R_4$$

$$R_4(x) = \frac{f^{(5)}(a)}{5!} x^5$$

$$\ln(1+x) \to \frac{1}{1+x} = (1+x)^{-1} \Rightarrow -1(1+x)^{-2} \Rightarrow 2(1+x)^{-3} \Rightarrow -6(1+x)^{-4}$$

$$\Rightarrow 24(1+x)^{-5} = \frac{24}{(1+x)^5}$$

$$\text{Max} \left|\frac{24}{(1+x)^5}\right| \text{ on } [-1/2, 1/2]$$

$$\frac{24}{(3/2)^5} = 24 \cdot \frac{2^5}{3^5} = \frac{256}{81}$$

$$\frac{256/81}{120} x^5 \Rightarrow \left|\frac{32}{1215}(\pm 1/2)^5\right| = \left|\frac{32}{1215}\left(\frac{\pm 1}{32}\right)\right| = \frac{1}{1215}$$

$R_4 < \frac{1}{1215} \approx 8.23 \times 10^{-4}$ on the interval $[0, 1/2]$

$$\left|\frac{24}{(1/2)^5}\right| = 24 \cdot 32 = 768$$

$\frac{768}{120} x^5 \Rightarrow \left|\frac{32}{5}(\pm 1/2)^5\right| = \frac{1}{5}$ this is the larger endpoint

$\boxed{\text{max error on } [-1/2, 1/2] \text{ is } 1/5}$