

# Homogeneous Solutions & Nullspaces

①

$$1a. \begin{cases} x_1 + 2x_2 - 3x_3 = 5 \\ 2x_1 + x_2 - 3x_3 = 13 \\ -x_1 + x_2 = -8 \end{cases}$$

$$\text{rref} \Rightarrow \left[ \begin{array}{ccc|c} 1 & 0 & -1 & 7 \\ 0 & 1 & -1 & -1 \\ 0 & 0 & 0 & 0 \end{array} \right]$$

$x_3$  free

$$x_1 - x_3 = 7 \Rightarrow x_1 = 7 + x_3$$

$$x_2 - x_3 = -1 \Rightarrow x_2 = -1 + x_3$$

$$x_3 = x_3$$

$$\Rightarrow \vec{x} = \begin{bmatrix} 7 \\ -1 \\ 0 \end{bmatrix} + t \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

$$b. \begin{cases} 2x_1 + 2x_2 + 4x_3 = 0 \\ -4x_1 - 4x_2 - 8x_3 = 0 \\ -3x_2 - 3x_3 = 0 \end{cases}$$

$$\text{rref} \Rightarrow \left[ \begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array} \right]$$

$x_3 = t$

$x_3$  is free

$$x_1 + x_3 = 0 \Rightarrow x_1 = -x_3$$

$$x_2 + x_3 = 0 \Rightarrow x_2 = -x_3$$

$$x_3 = x_3$$

$$\Rightarrow \vec{x} = t \begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix} \quad t = x_3$$

$$c. \begin{cases} 5x_1 - 3x_2 + 2x_3 = 0 \\ -3x_1 - 4x_2 + 2x_3 = 0 \end{cases}$$

$$\text{rref} \Rightarrow \left[ \begin{array}{ccc|c} 1 & 0 & 2/29 & 0 \\ 0 & 1 & -16/29 & 0 \end{array} \right]$$

$x_3$  is free

$$x_1 + 2/29 x_3 = 0 \Rightarrow x_1 = -2/29 x_3$$

$$x_2 - 16/29 x_3 = 0 \Rightarrow x_2 = 16/29 x_3$$

$$x_3 = x_3$$

$$\Rightarrow \vec{x} = t \begin{bmatrix} -2 \\ 16 \\ 29 \end{bmatrix}$$

$$t = x_3/29$$

$$d. \begin{cases} x_1 + 2x_2 - 3x_3 + x_4 - x_5 = 1 \\ 2x_1 - 2x_2 + 5x_3 - 2x_4 - x_5 = 6 \\ x_1 + 4x_2 - 4x_3 + x_4 + 2x_5 = 3 \\ -3x_1 + 5x_2 - 9x_3 - 8x_4 + 5x_5 = -7 \end{cases}$$

$$\text{rref} \Rightarrow \left[ \begin{array}{ccccc|c} 1 & 0 & 0 & 0 & -3/2 & 3/2 \\ 0 & 1 & 0 & 0 & 2/10 & 29/18 \\ 0 & 0 & 1 & 0 & 6/5 & 1/9 \\ 0 & 0 & 0 & 1 & -1/10 & -1/18 \end{array} \right]$$

$x_5$  is free

$$x_1 - 3/2 x_5 = 3/2 \Rightarrow x_1 = 3/2 + 3/2 x_5$$

$$x_2 + 2/10 x_5 = 29/18 \Rightarrow x_2 = 29/18 - 2/10 x_5$$

$$x_3 + 6/5 x_5 = 1/9 \Rightarrow x_3 = 1/9 - 6/5 x_5$$

$$x_4 - 1/10 x_5 = -1/18 \Rightarrow x_4 = -1/18 + 1/10 x_5$$

$$x_5 = x_5$$

$$\Rightarrow \vec{x} = \begin{bmatrix} 3/2 \\ 29/18 \\ 1/9 \\ -1/18 \\ 0 \end{bmatrix} + t \begin{bmatrix} 15 \\ -21 \\ -12 \\ 1 \\ 10 \end{bmatrix}$$

$$t = \frac{x_5}{10}$$

2.a.  $A = \begin{bmatrix} 1 & -3 & -8 & 5 \\ 0 & 1 & 2 & -4 \end{bmatrix}$  rref  $\Rightarrow \begin{bmatrix} 1 & 0 & -2 & -7 \\ 0 & 1 & 2 & -4 \end{bmatrix}$  (2)

$x_3, x_4$  free

$x_1 - 2x_3 - 7x_4 = 0 \Rightarrow x_1 = 2x_3 + 7x_4$

$x_2 + 2x_3 - 4x_4 = 0 \Rightarrow x_2 = -2x_3 + 4x_4$

$x_3 = x_3$

$x_4 = x_4$

$\Rightarrow \vec{x} = t \begin{bmatrix} 2 \\ -2 \\ 1 \\ 0 \end{bmatrix} + s \begin{bmatrix} 7 \\ 4 \\ 0 \\ 1 \end{bmatrix}$

$t = x_3, s = x_4$

$\text{Nul } A = \left\{ \begin{bmatrix} 2 \\ -2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 7 \\ 4 \\ 0 \\ 1 \end{bmatrix} \right\}$

b.  $A = \begin{bmatrix} 1 & -4 & -2 & 0 & 3 & -5 \\ 0 & 0 & 1 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 1 & 4 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$  rref  $\Rightarrow \begin{bmatrix} 1 & -4 & 0 & 0 & 0 & -19 \\ 0 & 0 & 1 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 1 & 4 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$

$x_2, x_4, x_6$  free

$x_1 - 4x_2 - 19x_6 = 0 \Rightarrow x_1 = 4x_2 + 19x_6$

$x_2 = x_2$

$x_3 - x_6 = 0 \Rightarrow$

$x_3 = x_6$

$x_6$

$t = x_2$

$x_4 = x_4$

$s = x_4$

$x_5 + 4x_6 = 0 \Rightarrow$

$x_5 = -4x_6$

$-4x_6$

$r = x_6$

$x_6 = x_6$

$x_6$

$\Rightarrow \vec{x} = t \begin{bmatrix} 4 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} + s \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} + r \begin{bmatrix} 19 \\ 0 \\ 0 \\ 0 \\ -4 \\ 1 \end{bmatrix}$

$\text{Nul } A = \left\{ \begin{bmatrix} 4 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 19 \\ 0 \\ 0 \\ 0 \\ -4 \\ 1 \end{bmatrix} \right\}$

c.  $A = \begin{bmatrix} 10 & -8 & -2 & -2 \\ 0 & 2 & 2 & -2 \\ 1 & -1 & 6 & 0 \\ 1 & 1 & 0 & -2 \end{bmatrix}$  rref  $\Rightarrow \begin{bmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

$x_4$  is free

$x_1 - x_4 = 0 \Rightarrow x_1 = x_4$

$x_2 - x_4 = 0 \Rightarrow x_2 = x_4$

$x_3 = 0$

$x_3 = 0$

$x_4 = x_4$

$\Rightarrow \vec{x} = t \begin{bmatrix} 1 \\ 1 \\ 0 \\ 1 \end{bmatrix} \quad t = x_4$

$\text{Nul } A = \left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \\ 1 \end{bmatrix} \right\}$

2d.  $A = \begin{bmatrix} 5 & -2 & 3 \\ -1 & 0 & -1 \\ 0 & -2 & -2 \\ 5 & 7 & 2 \end{bmatrix}$  rref  $\Rightarrow \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \Rightarrow \begin{matrix} x_1 = 0 \\ x_2 = 0 \\ x_3 = 0 \end{matrix} \rightarrow X = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

$\text{Nul } A = \{ \vec{0} \}$

3a.  $T\left(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}\right) = \begin{bmatrix} 2x_1 - x_3 \\ -x_1 + 3x_2 + x_3 \end{bmatrix}$   $A = \begin{bmatrix} 2 & 0 & -1 \\ -1 & 3 & 1 \end{bmatrix}$  rref  $\Rightarrow$

$\begin{bmatrix} 1 & 0 & -1/2 \\ 0 & 1 & 1/6 \end{bmatrix}$   $x_3$  free  
 $x_1 - 1/2 x_3 = 0 \Rightarrow x_1 = 1/2 x_3$   
 $x_2 + 1/6 x_3 = 0 \Rightarrow x_2 = -1/6 x_3$   
 $x_3 = x_3$   
 $t = \frac{x_3}{6}$   
 $X = \begin{bmatrix} 3 \\ -1 \\ 6 \end{bmatrix}$

$\text{Ker } A = \left\{ \begin{bmatrix} 3 \\ -1 \\ 6 \end{bmatrix} \right\}$

b.  $T\left(\begin{bmatrix} r \\ s \\ t \end{bmatrix}\right) = \begin{bmatrix} 2s + t \\ r - s + 2t \\ 3r + s \\ 2r - s - t \end{bmatrix}$   $A = \begin{bmatrix} 0 & 2 & 1 \\ 1 & -1 & 2 \\ 3 & 1 & 0 \\ 2 & -1 & -1 \end{bmatrix}$  rref  $\Rightarrow \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$

$x_1 = 0$   
 $x_2 = 0$   
 $x_3 = 0$   
 $\Rightarrow \vec{x} = \vec{0}$   $\text{Ker } A = \left\{ \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \right\}$

c.  $A = \begin{bmatrix} 1 & 2 & -4 & 3 & -2 & 6 & 0 \\ 0 & 0 & 0 & 1 & 0 & -3 & 7 \\ 0 & 0 & 0 & 0 & 1 & 4 & -2 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$  rref  $\Rightarrow \begin{bmatrix} 1 & 2 & -4 & 0 & 0 & 23 & 0 \\ 0 & 0 & 0 & 1 & 0 & -3 & 0 \\ 0 & 0 & 0 & 0 & 1 & 4 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$   
 $x_2, x_3, x_6$  free

$x_1 + 2x_2 - 4x_3 + 23x_6 = 0 \Rightarrow x_1 = -2x_2 - 4x_3 - 23x_6$   
 $x_2 = x_2$   
 $x_3 = x_3$   
 $x_4 - 3x_6 = 0 \Rightarrow x_4 = 3x_6$   
 $x_5 + 4x_6 = 0 \Rightarrow x_5 = -4x_6$   
 $x_6 = x_6$   
 $x_7 = 0 \Rightarrow x_7 = 0$   
 $t = x_2$   
 $s = x_3$   
 $r = x_6$

$\vec{x} = t \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} + s \begin{bmatrix} -4 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} + r \begin{bmatrix} -23 \\ 0 \\ 0 \\ 3 \\ -4 \\ 1 \\ 0 \end{bmatrix}$   $\text{Ker } A = \left\{ \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -4 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -23 \\ 0 \\ 0 \\ 3 \\ -4 \\ 1 \\ 0 \end{bmatrix} \right\}$

$$3d. A = \begin{bmatrix} 1 & 1 & -1 \\ 0 & 2 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

rref  $\Rightarrow$

$$\begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$x_1 - x_3 = 0$$

$$x_2 = 0$$

(4)

$$\Rightarrow \begin{aligned} x_1 &= x_3 \\ x_2 &= 0 \\ x_3 &= x_3 \end{aligned}$$

$$x_3 = t \Rightarrow$$

$$\vec{x} = t \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$$

$x_3$  free

$$\text{Ker } A = \left\{ \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \right\}$$