Instructions: Show all work. You may **not** use a calculator on this portion of the exam. **Give exact answers** (yes, that means fractions, square roots and exponentials, and not decimals). Reduce as much as possible. Be sure to complete all parts of each question. Provide explanations where requested. When you are finished with this portion of exam, get Part II.

1. For each of the matrices shown below, find the eigenvalues and eigenvectors of the matrix. (6 points each)

a. $A = \begin{bmatrix} -4 & 2 \\ 6 & 7 \end{bmatrix}$

$(-4-\lambda)(7-\lambda) - 12 = 0$

$\lambda^2 - 3\lambda - 28 - 12 = 0$

$\lambda^2 - 3\lambda - 40 = 0$

$(\lambda+5)(\lambda-8) = 0$

$\lambda_1 = -5 \quad \lambda_2 = 8$

$\lambda_1 \Rightarrow \begin{bmatrix} 1 & 2 \\ 6 & 12 \end{bmatrix}$

$x_1 = -2x_2$
$x_2 = x_2$

$\vec{v_1} = \begin{bmatrix} -2 \\ 1 \end{bmatrix}$

$\lambda_2 \Rightarrow \begin{bmatrix} -12 & 2 \\ 6 & -1 \end{bmatrix}$

$6x_1 = x_2 \rightarrow x_1 = \frac{1}{6}x_2$
$x_2 = x_2$

$\vec{v_2} = \begin{bmatrix} 1 \\ 6 \end{bmatrix}$

b. $B = \begin{bmatrix} -8 & -1 \\ 7 & 4 \end{bmatrix}$

$(-8-\lambda)(4-\lambda) + 7 = 0$

$-32 + 4\lambda + \lambda^2 + 7 = 0$

$\lambda^2 + 4\lambda - 25 = 0$

$\lambda = \dfrac{-4 \pm \sqrt{16+100}}{2} = \dfrac{-4 \pm \sqrt{116}}{2} = \dfrac{-4 \pm 2\sqrt{29}}{2} = -2 \pm \sqrt{29}$

$\lambda_1 \Rightarrow \begin{bmatrix} -8(-2+\sqrt{29}) & -1 \\ 7 & 4-(-2+\sqrt{29}) \end{bmatrix} = \begin{bmatrix} -6-\sqrt{29} & -1 \\ 7 & 6-\sqrt{29} \end{bmatrix}$

$7x_1 = (-6+\sqrt{29})x_2$
$x_2 = x_2$

$\vec{v_1} = \begin{bmatrix} -6+\sqrt{29} \\ 7 \end{bmatrix} \quad \vec{v_2} = \begin{bmatrix} -6-\sqrt{29} \\ 7 \end{bmatrix}$

2. The weather in a particular town in the Pacific northwest is classified as either good, indifferent or bad. If the weather is good on a particular day, the probability it will remain good the next day is only 11%, and there is a 45% chance it will be indifferent. If the weather is indifferent on a particular day, there is a 20% chance it will become good, and a 42% chance it will become bad. If the weather is bad on a particular day, there is a 48% chance it will be bad the next day, and a 21% chance it will become indifferent. Construct the stochastic matrix that models this problem. (5 points)

$$
\begin{array}{c}
 & G & I & B \\
\begin{array}{c} G \\ I \\ B \end{array} &
\left[\begin{array}{ccc}
.11 & .20 & .31 \\
.45 & .38 & .21 \\
.44 & .42 & .48
\end{array}\right]
\end{array}
$$

3. Determine if each statement is True or False. For each of the questions, assume that A is $n \times n$. (2 points each)

a. (T) F If λ is an eigenvalue of A, then λ^3 is an eigenvalue of A^3.

b. (T) F If zero is not an eigenvalue of A, then the determinant of A is non-zero.

c. (T) F If $\vec{v}$ is an eigenvector of A, then $\vec{v}$ is also an eigenvector of e^A.

d. T (F) If A is diagonalizable, then A is invertible. λ could be 0

e. T (F) Row operations on a matrix do not change its eigenvalues.

f. (T) F Similar matrices always have the same eigenvalues.

g. T (F) The dimension of the eigenspace of an $n \times n$ matrix is always n.

h. (T) F The vector $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ is not an eigenvector of $\begin{bmatrix} 7 & -7 \\ 6 & 4 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 7-7 \\ 6+4 \end{bmatrix} = \begin{bmatrix} 0 \\ 10 \end{bmatrix}$

i. T (F) A 7x7 matrix has three eigenvalues, one eigenspace is one dimensional, one eigenspace is two dimensional, and one eigenspace is three dimensional; therefore this matrix is diagonalizable. $1+2+3=6$

j. (T) F Stochastic matrices, regardless of their size, always have at least one real eigenvector.

k.   T   (**F**)   The real eigenvalues of a discrete dynamical system must always both attract or repel from the origin.

l.   T   (**F**)   In a system of ODEs, the magnitude of λ determines whether a complex eigenvalues causes the origin to repel or attract.

m.   (**T**)   F   The dot product or scalar product is one type of inner product.

n.   (**T**)   F   To be an inner product space, a space needs to be both vector space, and have a particular inner product defined on it.

o.   (**T**)   F   Normalizing a vector refers to making a vector pointing in a particular direction have components that satisfy certain conditions.

p.   T   (**F**)   The distance between two points (vectors) in R^n is defined to be $\|\vec{u}\| - \|\vec{v}\|$.

4. Find the equilibrium vector of the matrix $P = \begin{bmatrix} .1 & .6 \\ .9 & .4 \end{bmatrix}$ algebraically. Be sure to properly normalize the vector. (6 points)

$$P - I = \begin{bmatrix} -.9 & .6 \\ .9 & -.6 \end{bmatrix}$$

$$.9x_1 = .6x_2$$
$$x_1 = \frac{6}{9}x_2$$
$$x_2 = x_2$$

$$\vec{x} = \begin{bmatrix} 6 \\ 9 \end{bmatrix} \implies 6+9=15$$

$$\frac{6}{15} = \frac{2}{5}$$
$$\frac{9}{15} = \frac{3}{5}$$

$$\vec{q} = \begin{bmatrix} 2/5 \\ 3/5 \end{bmatrix}$$

5. For each of the situations below, determine the properties of the discrete dynamical system. Is the origin an attractor, a repeller, or a saddle point? Sketch the eigenvalues on the graphs provided (if they are real) and plot some sample trajectories. (4 points each)
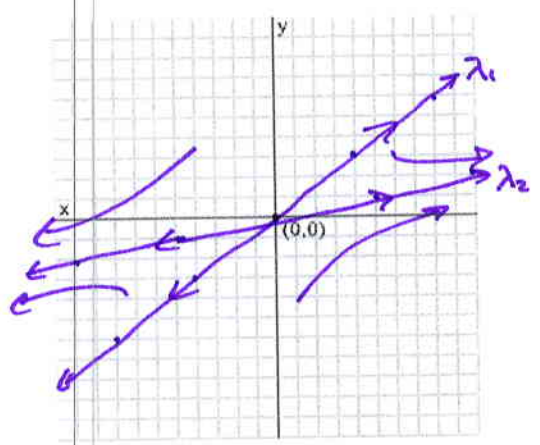
   a. $\lambda_1 = 0.9, \lambda_2 = 0.4, \vec{v}_1 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}, \vec{v}_2 = \begin{bmatrix} -1 \\ 4 \end{bmatrix}$.
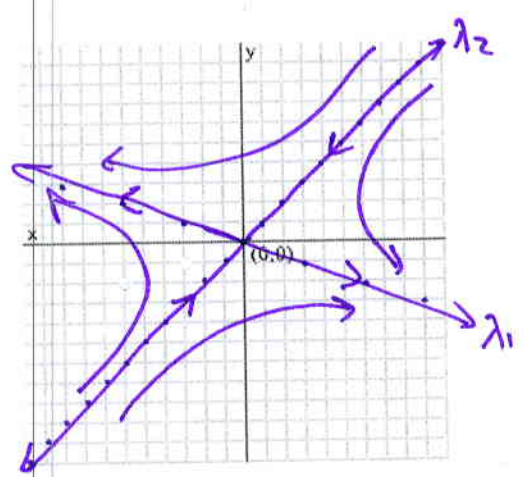
   attracts to origin
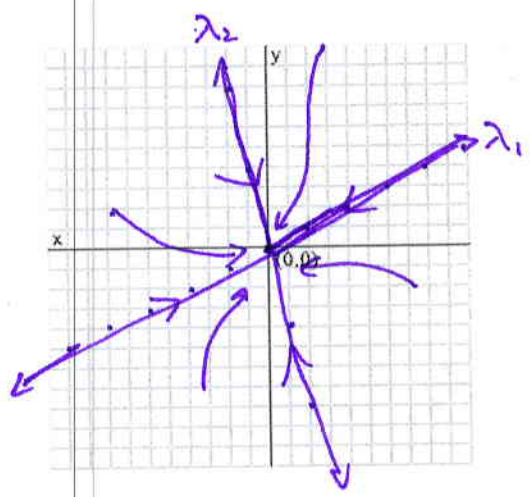
   b. $\lambda_1 = 1.2, \lambda_2 = -3, \vec{v}_1 = \begin{bmatrix} 4 \\ 3 \end{bmatrix}, \vec{v}_2 = \begin{bmatrix} 5 \\ 1 \end{bmatrix}$.

   Origin repels

   c. $\lambda_1 = -2.1, \lambda_2 = 0.4, \vec{v}_1 = \begin{bmatrix} -3 \\ 1 \end{bmatrix}, \vec{v}_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$.
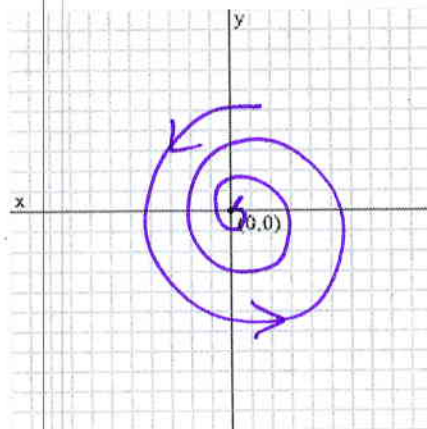
   Origin is a saddle point

d. $\lambda_1 = 0.5 + 0.6i, \lambda_2 = 0.5 - 0.6i.$

$\sqrt{.5^2 + .6^2} \approx .78$

Origin attracts

$|\lambda| < 1$

Spirals in

6. For the vectors $\vec{u} = \begin{bmatrix} 1 \\ 9 \\ -3 \end{bmatrix}, \vec{v} = \begin{bmatrix} 4 \\ 5 \\ -1 \end{bmatrix}$, find the following: (3 points each)

a. $\|\vec{v}\|$

$\sqrt{16 + 25 + 1} = \sqrt{42}$

b. A unit vector in the direction of $\vec{u}$.

$\|\hat{u}\| = \sqrt{1 + 81 + 9} = \sqrt{91}$

$\hat{u} = \begin{bmatrix} 1/\sqrt{91} \\ 9/\sqrt{91} \\ -3/\sqrt{91} \end{bmatrix}$

c. $\vec{u} \cdot \vec{v}$

$4 + 45 + 3 = 52$

d. Are the two vectors orthogonal? Why or why not?

they are not since the dot product is not zero.

Instructions: Show all work. You **may** use a calculator on this portion of the exam. To show work on calculator problems, show the commands you used, and the resulting matrices. **Give exact answers** (yes, that means fractions, square roots and exponentials, and not decimals) unless specifically directed to give a decimal answer. This will require some operations to be done by hand even if not specifically directed to. Be sure to complete all parts of each question. Provide explanations where requested.

1. Find the eigenvalues and eigenvectors for the matrix $A = \begin{bmatrix} 4 & 0 & -1 \\ -1 & 0 & 4 \\ 0 & 2 & 3 \end{bmatrix}$. (9 points)

$\lambda = 5$ $\begin{bmatrix} -1 & 0 & -1 \\ -1 & -5 & 4 \\ 0 & 2 & -2 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}$ $\begin{array}{l} x_1 = -x_3 \\ x_2 = x_3 \\ x_3 = x_3 \end{array} \Rightarrow \vec{v_1} = \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix}$

$\lambda = 1 + \sqrt{7}$ $\begin{bmatrix} 4-(1+\sqrt{7}) & 0 & -1 \\ -1 & -1-\sqrt{7} & 4 \\ 0 & 2 & 3-(1+\sqrt{7}) \end{bmatrix} \Rightarrow \begin{bmatrix} 3-\sqrt{7} & 0 & -1 \\ -1 & -1-\sqrt{7} & 4 \\ 0 & 2 & 2-\sqrt{7} \end{bmatrix}$

$(3-\sqrt{7})x_1 = x_3$
$2x_2 = (-2+\sqrt{7})x_3 \Rightarrow$
$x_3 = x_3$

$x_1 = \frac{3+\sqrt{7}}{2} x_3$
$x_2 = \frac{-2+\sqrt{7}}{2} x_3$
$x_3 = x_3$

$\frac{1}{3\sqrt{7}} \cdot \frac{3+\sqrt{7}}{3+\sqrt{7}} = \frac{3+\sqrt{7}}{9-7} = \frac{3+\sqrt{7}}{2}$

$\vec{v_2} = \begin{bmatrix} 3+\sqrt{7} \\ -2+\sqrt{7} \\ 2 \end{bmatrix}$ $\vec{v_3} = \begin{bmatrix} 3-\sqrt{7} \\ -2-\sqrt{7} \\ 2 \end{bmatrix}$

start →

$(4-\lambda)[-\lambda(3-\lambda)-8] - 0 - 1(-2+0) = 0$

$(4-\lambda)(\lambda^2-3\lambda-8) + 2 = 0$

$4\lambda^2 - 12\lambda - 32 - \lambda^3 + 3\lambda^2 + 8\lambda + 2 = 0$

$-\lambda^3 + 7\lambda^2 - 4\lambda - 30 = 0$

$\lambda^3 - 7\lambda^2 + 4\lambda + 30 = 0$ *from graph, one zero is at $\lambda = 5$*

$\begin{array}{r} \lambda^2 - 2\lambda - 6 \\ \lambda-5 \overline{) \lambda^3 - 7\lambda^2 + 4\lambda + 30} \\ -\lambda^3 + 5\lambda^2 \\ \hline -2\lambda^2 + 4\lambda + 30 \\ +2\lambda^2 \mp 10\lambda \\ \hline -6\lambda + 30 \\ -6\lambda + 30 \\ \hline 0 \end{array}$

$(\lambda-5)(\lambda^2-2\lambda-6)$ $\lambda = 5$

$\lambda = \frac{2 \pm \sqrt{4+24}}{2} = \frac{2 \pm \sqrt{28}}{2} = \frac{2 \pm 2\sqrt{7}}{2}$

$= 1 \pm \sqrt{7}$

2. Find the similarity transformation for the matrix $B = \begin{bmatrix} 4 & -2 \\ 1 & 6 \end{bmatrix}$ that converts this matrix into a similarity rotation matrix. Then use that matrix to find the angle of rotation. Give your angle in radians rounded to 4 decimal places, or in degrees rounded to one decimal place. (10 points)

$(4-\lambda)(6-\lambda) + 2 = 0$

$24 - 10\lambda + \lambda^2 + 2 = 0$

$\lambda^2 - 10\lambda + 26 = 0$

$\lambda = \dfrac{10 \pm \sqrt{100 - 104}}{2} =$

$\dfrac{10 \pm 2i}{2} = 5 \pm i$

$C = \begin{bmatrix} 5 & -1 \\ 1 & 5 \end{bmatrix} \quad P = \begin{bmatrix} -1 & -1 \\ 1 & 0 \end{bmatrix}$

$\begin{bmatrix} 4-(5+i) & -2 \\ 1 & 6-(5+i) \end{bmatrix} = \begin{bmatrix} -1-i & -2 \\ 1 & 1-i \end{bmatrix}$

$x_1 = (-1+i)x_2$
$x_2 = x_2$

$\vec{v}_1 = \begin{bmatrix} -1+i \\ 1 \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \end{bmatrix}i$

$r^2 = 25 + 1 = 26$

$C = \sqrt{26}\begin{bmatrix} 5/\sqrt{26} & -1/\sqrt{26} \\ 1/\sqrt{26} & 5/\sqrt{26} \end{bmatrix}$

$\cos^{-1}(5/\sqrt{26}) = .1973956\cdots \boxed{.1974}$

or $= 11.3°$

3. Use your calculator to find the equilibrium vector of the stochastic matrix $P = \begin{bmatrix} .6 & .4 & .1 & .2 \\ .1 & .6 & .2 & .3 \\ .1 & 0 & .5 & .1 \\ .2 & 0 & .2 & .4 \end{bmatrix}$. Explain the steps you took to obtain the vector. Then demonstrate that it is the correct equilibrium vector by multiplying by P. (6 points)

$P^{\wedge}60 \quad (\approx \text{limit as } k \to \infty \text{ for } P^k)$

$\begin{bmatrix} 7/17 & 7/17 & 7/17 & 7/17 \\ 5/17 & 5/17 & 5/17 & 5/17 \\ 2/17 & 2/17 & 2/17 & 2/17 \\ 3/17 & 3/17 & 3/17 & 3/17 \end{bmatrix}$

$\vec{q} = \begin{bmatrix} 7/17 \\ 5/17 \\ 2/17 \\ 3/17 \end{bmatrix}$

yes, $P\vec{q}$ does $= \vec{q}$

4. Does the matrix in #3 have enough communication between states to have only one equilibrium vector? Does the matrix have any absorbing states? Explain your reasoning. (5 points)

yes it does (have enough communication)

if you start in any state, there is a path to any other state.

there are no absorbing states (no entries = 1)

5. Solve the discrete dynamical system given by $\vec{x}_{k+1} = \begin{bmatrix} 0.3 & 0.4 \\ -0.3 & 1.1 \end{bmatrix} \vec{x}_k$. Find the eigenvalues and eigenvectors. (8 points)

$$(.3 - \lambda)(1.1 - \lambda) + .12 = 0$$

$$.33 - 1.4\lambda + \lambda^2 + .12 = 0$$

$$\lambda^2 - 1.4\lambda + .45 = 0$$

$$(\lambda - .9)(\lambda - .5) = 0$$

$$\lambda_1 = .9, \quad \lambda_2 = .5$$

$$\lambda_1 \Rightarrow \begin{bmatrix} -.6 & .4 \\ -.3 & .2 \end{bmatrix}$$

$$-.3x_1 = -.2x_2$$
$$x_1 = \frac{2}{3}x_2$$
$$x_2 = x_2$$

$$\vec{v_1} = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$$

$$\lambda_2 \Rightarrow \begin{bmatrix} -.2 & .4 \\ -.3 & .6 \end{bmatrix}$$

$$-.2x_1 = -.4x_2$$
$$x_1 = 2x_2$$
$$x_2 = x_2$$

$$\vec{v_2} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

6. Solve the system of ODEs given by $\vec{x}' = \begin{bmatrix} 1 & -2 \\ 3 & -4 \end{bmatrix} \vec{x}$. Sketch a graph of the eigenvectors and plot some sample trajectories. Is the origin an attractor, a repeller or a saddle point? Give your final solution in exact form with e. (10 points)

$(1-\lambda)(-4-\lambda) + 6 = 0$

$-4 + 3\lambda + \lambda^2 + 6 = 0$

$\lambda^2 + 3\lambda + 2 = 0$

$(\lambda + 2)(\lambda + 1) = 0$

$\lambda = -2, -1$

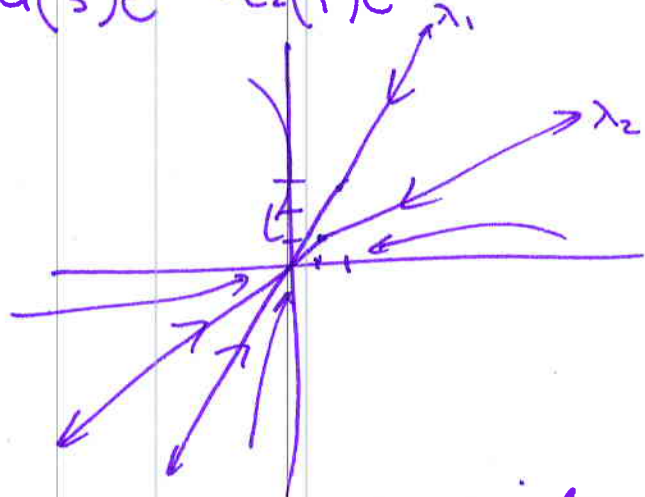$\lambda_2 = -1 \Rightarrow \begin{bmatrix} 2 & -2 \\ 3 & -3 \end{bmatrix}$

$2x_1 = 2x_2$
$x_2 = x_2$

$\vec{v_2} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

$\vec{x} = c_1 \begin{bmatrix} 2 \\ 3 \end{bmatrix} e^{-2t} + c_2 \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{-t}$

$\lambda_1 = -2 \Rightarrow \begin{bmatrix} 3 & -2 \\ 3 & -2 \end{bmatrix}$

$3x_1 = 2x_2$
$x_1 = \frac{2}{3} x_2$
$x_2 = x_2$

$\vec{v_1} = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$

Origin attracts in both directions

7. The vectors $\vec{u} = \begin{bmatrix} 2 \\ -3 \\ 1 \\ 0 \end{bmatrix}$, $\vec{v} = \begin{bmatrix} 3 \\ 2 \\ 0 \\ 1 \end{bmatrix}$ are orthogonal to each other. Find a vector $\vec{w} = \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix}$ orthogonal to both. (6 points)

$\begin{bmatrix} 2 & -3 & 1 & 0 \\ 3 & 2 & 0 & 1 \end{bmatrix} \Rightarrow \text{rref} \Rightarrow \begin{bmatrix} 1 & 0 & 2/13 & 3/13 \\ 0 & 1 & 3/13 & 2/13 \end{bmatrix}$

free

$x_1 = -2/13 x_3 - 3/13 x_4$
$x_2 = -3/13 x_3 - 2/13 x_4$
$x_3 = x_3$
$x_4 = x_4$

$\Rightarrow \begin{bmatrix} -2 \\ -3 \\ 13 \\ 0 \end{bmatrix}$ or $\begin{bmatrix} -3 \\ -2 \\ 0 \\ 13 \end{bmatrix}$

answers may vary

8. Determine if the polynomials $p(t) = 2 - t, q(t) = t^2 + 2t$ are orthogonal under the inner product $\langle f | g \rangle \int_{-2}^{2} f(t)g(t)dt$. (6 points)

$$\int_{-2}^{2} (2-t)(t^2+2t)\,dt$$

$$= \int_{-2}^{2} 2t^2 + 4t - t^3 - 2t^2 \, dt$$

$$= \int_{-2}^{2} 4t - t^3 \, dt \qquad\qquad = 0$$

these terms are odd so

yes, they are orthogonal