MAT 201, Exam #2 (Make-up Points), Spring 2016 Name KEY

Instructions: You can solve these problems by hand or using technology, or some combination—this last is the most likely scenario. (You may use MatLab, for instance.) You should provide a graph that shows surfaces or curves of intersection, or paths. To obtain the graphs, you can use MatLab or another software program (free or otherwise). Each revised problem is worth 4 points.

1. Use Stokes' Theorem to evaluate $\int_C \vec{F}\cdot d\vec{r}$ for $\vec{F}(x,y,z) = z\hat{\imath} + xy\hat{\jmath} + 4x\hat{k}$ where C is the intersection of the plane $y+z=5$ and $x^2+y^2=16$ oriented counterclockwise from above.

$G = y+z-5$

$$\vec{\nabla}\times\vec{F} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ z & xy & 4x \end{vmatrix} = (0-0)\hat{\imath} + (4-1)\hat{\jmath} + (y-0) = 0\hat{\imath} - 3\hat{\jmath} + y\hat{k}$$

$\vec{n} = \nabla G = \langle 0,1,1\rangle$ $\quad (\vec{\nabla}\times\vec{F})\cdot\vec{N} = 0 - 3 + y = y-3$

$$\iint_R y-3\,dA \Rightarrow \int_0^{2\pi}\int_0^4 (r\sin\theta - 3)\,r\,dr\,d\theta = \int_0^{2\pi}\int_0^4 r^2\sin\theta - 3r\,dr\,d\theta =$$

$$\int_0^{2\pi} \left[\tfrac{1}{3}r^3\sin\theta - \tfrac{3}{2}r^2\right]_0^4 d\theta = \int_0^{2\pi} \tfrac{64}{3}\sin\theta - 24\,d\theta = -\tfrac{64}{3}\cos\theta - 24\theta\Big|_0^{2\pi}$$

$$\cancel{-\tfrac{64}{3}(1) - \left(-\tfrac{64}{3}(1)\right)} - 24(2\pi - 0) = \boxed{-48\pi}$$

2. Evaluate $\int_C \vec{F}\cdot d\vec{r}$ for $\vec{F}(x,y) = 3y\hat{\imath} + 7x\hat{\jmath}$ over the ellipse give by $x^2 + 4y^2 = 12$ [Recall: the area of an ellipse is $A = \pi ab$.]

12

$$\frac{x^2}{12} + \frac{y^2}{3} = 1$$

$a^2 = 12 \Rightarrow a = \sqrt{12}$

$b^2 = 3 \Rightarrow b = \sqrt{3}$

$$\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} = 7 - 3 = 4$$

$$\iint_R 4\,dA = \pi(\sqrt{12})(\sqrt{3})\cdot 4 =$$

$$4\pi(\sqrt{36}) = 4\pi * 6$$

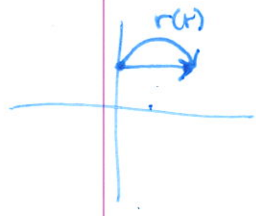
$$\boxed{= 24\pi}$$

3. Use the fact that $\vec{F}(x,y) = (xy^2 + ye^{xy})\hat{\imath} + (x^2y + xe^{xy})\hat{\jmath}$ is conservative to evaluate $\int_C \vec{F} \cdot d\vec{r}$ on $C: \vec{r}(t) = \left[t + \sin\left(\frac{\pi}{2}t^2\right)\right]\hat{\imath} + \left[t + \cos\left(\frac{\pi}{2}t^2\right)\right]\hat{\jmath}, 0 \le t \le 1$. (Since this field is conservative, show the provided curve, and the straight-line path on the same graph.)

$\int M\,dx = \int xy^2 + ye^{xy}\,dx = \frac{1}{2}x^2y^2 + e^{xy} + f(y)$

$\int N\,dy = \int x^2y + xe^{xy}\,dy = \frac{1}{2}x^2y^2 + e^{xy} + g(x)$

$h(x,y) = \frac{1}{2}x^2y^2 + e^{xy}$ $(+c)$



endpoints $t=0$ $\quad 0+0=0\hat{\imath}$ $\quad (0,1)$

$0+1=1\hat{\jmath}$

$t=1$ $\quad 1+\sin(\pi/2) = 1+1 = 2$ $\quad (2,1)$

$1+\cos(\pi/2) = 1+0 = 1$

$\int_C F \cdot dr = h(2,1) - h(0,1) = \frac{1}{2}(2)^2(1)^2 + e^{(2)(1)} - \left(\frac{1}{2}(0)^2(1)^2 + e^{0(1)}\right) =$

$2 + e^2 - 1 = \boxed{e^2+1}$

4. Evaluate $\int_C \vec{F} \cdot d\vec{r}$ for $\vec{F}(x,y,z) = x\hat{\imath} - y\hat{\jmath} + xy\hat{k}$ over the curve $C: \vec{r}(t) = \cos 4t\,\hat{\imath} + \sin 4t\,\hat{\jmath} + t\hat{k}, 0 \le t \le \pi$. Be sure to sketch the curve.

$\int_0^\pi \langle \cos 4t, -\sin 4t, \cos 4t \sin 4t \rangle \cdot \langle -4\sin 4t, 4\cos 4t, 1 \rangle\,dt$

$= \int_0^\pi -4\sin 4t \cos 4t - 4\sin 4t \cos 4t + \cos 4t \sin t\,dt$

$= \int_0^\pi -7\cos 4t \sin 4t\,dt$

$u = \sin 4t$

$du = 4\cos 4t\,dt$

$\frac{1}{4}du = \cos 4t\,dt$

$\Rightarrow \int_0^0 -\frac{7}{4}u\,du = -\frac{7}{8}u^2 \Rightarrow$

$-\frac{7}{8}\sin^2 4t \Big|_0^\pi = 0 - 0 = \boxed{0}$

5. Evaluate $\int_C y\sec^2 x\,ds$ on the line segment from $(-1,2)$ to $(5,4)$.

$\langle 6,2\rangle$

$\vec{r}(t) = (6t-1)\hat{\imath} + (2t+2)\hat{\jmath} \quad 0\le t\le 1$

$r'(t) = 6\hat{\imath} + 2\hat{\jmath}$

$\int_0^1 (2t+2)(\sec^2(6t-1))\,dt \cdot 2\sqrt{10} \qquad \|r'(t)\| = \sqrt{36+4} = \sqrt{40} = 2\sqrt{10}$

$4\sqrt{10}\int_0^1 (t+1)\sec^2(6t-1)\,dt \qquad u = t+1 \quad dv = \sec^2(6t-1)$

$du = dt \quad v = \frac{1}{6}\tan(6t-1)$

$4\sqrt{10}\left[(t+1)\frac{1}{6}\tan(6t-1) - \int_0 \frac{1}{6}\tan(6t-1)\,dt\right.$

$4\sqrt{10}\left[\frac{1}{6}(t+1)\tan(6t-1) + \frac{1}{36}\ln|\cos(6t-1)|\right|_0^1 =$

$4\sqrt{10}\left[\frac{1}{6}(2)\tan(5) + \frac{1}{36}\ln|\cos(5)| - \frac{1}{6}(1)\underbrace{\tan(-1)}_{-\pi/4} - \frac{1}{36}\ln|\cos(-1)|\right|$

$\boxed{4\sqrt{10}\left[\frac{1}{3}\tan 5 + \frac{1}{36}\ln|\cos 5| + \frac{\pi}{24} - \frac{1}{36}\ln|\cos(-1)|\right]}$

6. Find the area of the region inside $r = 1+\cos\theta$ and outside $r = 3\cos\theta$.

$1+\cos\theta = 3\cos\theta$

$1 = 2\cos\theta$

$1/2 = \cos\theta$

$\theta = \pi/3, -\pi/3$

$2\left[\int_{\pi/3}^{\pi/2}\int_{3\cos\theta}^{1+\cos\theta} r\,dr\,d\theta + \int_{\pi/2}^{\pi}\int_0^{1+\cos\theta} r\,dr\,d\theta\right] = \cancel{2}\left[\int_{\pi/3}^{\pi/2}\cancel{\tfrac{1}{2}}\left[(1+\cos\theta)^2 - (3\cos\theta)^2\right]d\theta\right.$

$\left.+ \int_{\pi/2}^{\pi}\cancel{\tfrac{1}{2}}(1+\cos\theta)^2\,d\theta\right] = \int_{\pi/3}^{\pi/2} 1+2\cos\theta+\cos^2\theta - 9\cos^2\theta\,d\theta + \int_{\pi/2}^{\pi} 1+2\cos\theta+\cos^2\theta\,d\theta$

$= \int_{\pi/3}^{\pi/2} 1+2\cos\theta - 8\cos^2\theta\,d\theta + \int_{\pi/2}^{\pi} 1+2\cos\theta+\cos^2\theta\,d\theta =$

$= \int_{\pi/3}^{\pi/2} 1+2\cos\theta - 4 - 4\cos 2\theta\,d\theta + \int_{\pi/2}^{\pi} 1+2\cos\theta+\frac{1}{2}+\frac{1}{2}\cos 2\theta\,d\theta$

$= \left[-3\theta + 2\sin\theta - 2\sin 2\theta\right|_{\pi/3}^{\pi/2} + \frac{3}{2}\theta + 2\sin\theta + \frac{1}{4}\sin 2\theta\Big|_{\pi/2}^{\pi}$

$= -\cancel{\frac{3\pi}{2}} + \cancel{2} - 0 + \pi - \cancel{\sqrt{3}} + \cancel{\sqrt{3}} + \cancel{\frac{3\pi}{2}} + 0 + 0 - \frac{3\pi}{4} - \cancel{2} - 0 = \boxed{\pi/4}$

7. Set up the triple integral needed to find the volume enclosed by the surfaces $x = 2y^2 + 2z^2$ and $x = 12 - y^2 - z^2$. Find the volume of the enclosed area. [Hint: using a version of cylindrical coordinates would make this easier. Specify your substitutions.]

$y = r\cos\theta \quad y^2+z^2=r^2$

$z = r\sin\theta$

$x = x$

$x = 2r^2$

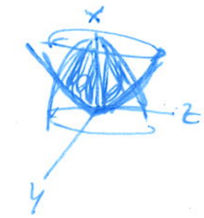
$x = 12-r^2$

$$\int_0^{2\pi}\int_0^{2}\int_{2r^2}^{12-r^2} r\,dx\,dr\,d\theta =$$

$$\int_0^{2\pi}\int_0^{2}(12-r^2-2r^2)r\,dr\,d\theta = \int_0^{2\pi}\int_0^{2} 12-3r^2)r\,dr\,d\theta$$

$$= \int_0^{2\pi}\int_0^{2} 12r-3r^3\,dr\,d\theta = \int_0^{2\pi} 6r^2-\frac{3}{4}r^4\Big|_0^2\,d\theta =$$

$$\int_0^{2\pi} 24-12\,d\theta = \int_0^{2\pi} 12\,d\theta = \boxed{24\pi}$$



$2r^2 = 12-r^2$

$3r^2 = 12$

$r^2 = 4$

$r = 2$

8. Convert the integral $\int_{-3}^{3}\int_{-\sqrt{9-x^2}}^{\sqrt{9-x^2}}\int_{3-\sqrt{9-x^2-y^2}}^{3+\sqrt{9-x^2-y^2}}(x^2+y^2+z^2)^{\frac{5}{2}}\,dz\,dy\,dx$ into spherical coordinates. Evaluate it.

$(\rho^2)^{5/2} = \rho^5$

$z = 3 \pm \sqrt{9-x^2-y^2}$

$(z-3)^2 = (\pm\sqrt{9-x^2-y^2})^2$

$x^2+y^2+(z-3)^2 = 9$

$x^2+y^2+z^2-6z = 0$

$\rho^2 - 6\rho\cos\varphi = 0$

$\rho^{\cancel{2}} = 6\cancel{\rho}\cos\varphi$

$\rho = 6\cos\varphi$

$$\int_0^{2\pi}\int_0^{\pi/2}\int_0^{6\cos\varphi} \rho^5\cdot\rho^2\sin\varphi\,d\rho\,d\varphi\,d\theta =$$

$$\int_0^{2\pi}\int_0^{\pi/2}\int_0^{6\cos\varphi} \rho^7\sin\varphi\,d\rho\,d\varphi\,d\theta =$$

$$\int_0^{2\pi}\int_0^{\pi/2} \frac{1}{8}(6\cos\varphi)^8\sin\varphi\,d\varphi\,d\theta =$$

$$\frac{6^8}{8}\int_0^{2\pi}\int_0^{\pi/2} \cos^8\varphi\sin\varphi\,d\varphi\,d\theta$$

$u = \cos\varphi$

$du = -\sin\varphi\,d\varphi$

$$-\frac{6^8}{8}\int_0^{2\pi}\frac{1}{9}\cos^9\varphi\Big|_0^{\pi/2}d\theta \Rightarrow -\frac{6^8}{72}[0-1] \Rightarrow \frac{6^6}{2}\int_0^{2\pi}d\theta = \frac{6^6}{2}\cdot 2\pi$$

$36\cdot 2$

$= 6^6\pi$

$= \boxed{46{,}656\pi}$