MAT 201, Exam #3, Spring 2016 Name KEY

Instructions: Show all work. Answers without work required to obtain the solution will not receive full credit. Some questions may contain multiple parts: be sure to answer all of them. Give exact answers unless specifically asked to estimate.

1. Set up an integral to find the length of the curve $\vec{r}(t) = 12t\hat{\imath} + 8t^{\frac{3}{2}}\hat{\jmath} + 3t^2\hat{k}$ on $0 \le t \le 1$. (6 points)

$r'(t) = 12\hat{\imath} + 12t^{1/2}\hat{\jmath} + 6t\hat{k}$

$\|r'(t)\| = \sqrt{144 + 144t + 36t^2}$

$= 6\sqrt{4 + 4t + t^2} = 6\sqrt{(2+t)^2} =$

$6(2+t) = 12 + 6t$

$\int_0^1 \|r'(t)\|\,dt = \int_0^1 12 + 6t\,dt$

2. Find the unit tangent vector for $\vec{r}(t) = 2\sqrt{t}\hat{\imath} + e^t\hat{\jmath} + e^{-t}\hat{k}$. (6 points)

$r'(t) = \frac{1}{\sqrt{t}}\hat{\imath} + e^t\hat{\jmath} + (-e^{-t})\hat{k}$ $\|r'(t)\| = \sqrt{\frac{1}{t} + e^{2t} + e^{-2t}}$

$T(t) = \dfrac{\frac{1}{\sqrt{t}}\hat{\imath} + e^t\hat{\jmath} - e^{-t}\hat{k}}{\sqrt{\frac{1}{t} + e^{2t} + e^{-2t}}}$

changed problem: $\hat{\imath}$ term $\sqrt{2}t$

$r'(t) = \sqrt{2}\hat{\imath} \cdots$

$\|r'(t)\| = \sqrt{2 + \cdots} = e^t + e^{-t}$

$T(t) = \dfrac{\sqrt{2}\hat{\imath} + e^t\hat{\jmath} + e^{-t}k}{e^t + e^{-t}}$

3. Find the unit normal vector for $\vec{r}(t) = 3t\hat{\imath} + 4\sin t\,\hat{\jmath} + 4\cos t\,\hat{k}$. Use it to write an equation of the normal line at the point $t = \pi$. (8 points)

$r'(t) = 3\hat{\imath} + 4\cos t\hat{\jmath} - 4\sin t\hat{k}$

$\|r'(t)\| = \sqrt{3^2 + 16\cos^2 t + 16\sin^2 t} = \sqrt{9 + 16(1)} = \sqrt{25} = 5$

$\boxed{T(t) = \frac{3}{5}\hat{\imath} + \frac{4}{5}\cos t\hat{\jmath} - \frac{4}{5}\sin t\hat{k}}$

$T'(t) = 0\hat{\imath} + -\frac{4}{5}\sin t\hat{\jmath} - \frac{4}{5}\cos t\hat{k}$

$\|T'(t)\| = \sqrt{\frac{16}{25}\sin^2 t + \frac{16}{25}\cos^2 t} = \sqrt{\frac{16}{25}} = \frac{4}{5}$

$N(t) = \frac{5}{4}\left(-\frac{4}{5}\sin t\hat{\jmath} - \frac{4}{5}\cos t\hat{k}\right) = \boxed{-\sin t\hat{\jmath} - \cos t\hat{k}}$

$N(\pi) = 0\hat{\jmath} - (-1)\hat{k} = \hat{k}$

$r(\pi) = 3\pi\hat{\imath} + 0\hat{\jmath} + (-4)\hat{k}$ $(3\pi, 0, -4)$ $\boxed{l(t) = 3\pi\hat{\imath} + 0\hat{\jmath} + (t-4)\hat{k}}$

4. Find the function of the curvature for the equation $y = xe^x$. Then use it to estimate the radius of curvature at the point $x = 1$. (8 points)

$y' = e^x + xe^x = (1+x)e^x$

$y'' = e^x + e^x + xe^x = (2+x)e^x$

$K = \dfrac{|(2+x)e^x|}{(1+(1+x)^2 e^{2x})^{3/2}}$

$K(1) = \dfrac{3e}{(1+4e^2)^{3/2}}$

$\dfrac{1}{K} \approx R = \dfrac{(1+4e^2)^{3/2}}{3e} \approx \boxed{20.71}$

5. Find an equation of the tangent plane to the function $z = \sqrt{xy}$ at the point $(1,1,1)$. (6 points)

$F = \sqrt{x}\sqrt{y} - z$

$\nabla F = \langle \frac{\sqrt{y}}{2\sqrt{x}}, \frac{\sqrt{x}}{2\sqrt{y}}, -1 \rangle \quad \nabla F(1,1,1) = \langle \frac{1}{2}, \frac{1}{2}, -1 \rangle$

$\boxed{\frac{1}{2}(x-1) + \frac{1}{2}(y-1) - 1(z-1) = 0}$

6. Find the linear approximation $L(x, y, z)$ to the function $f(x, y, z) = \sqrt{x^2 + y^2 + z^2}$ at $(3,2,6)$, 7 and use it to estimate the value of $\sqrt{3.02^2 + 1.97^2 + 5.99^2}$. (7 points)

$F = \sqrt{x^2+y^2+z^2} - w$

$\nabla F = \langle \frac{\cancel{2}x}{\cancel{2}\sqrt{x^2+y^2+z^2}}, \frac{y}{\sqrt{x^2+y^2+z^2}}, \frac{z}{\sqrt{x^2+y^2+z^2}}, -1 \rangle \quad \nabla F(3,2,6) = \langle \frac{3}{7}, \frac{2}{7}, \frac{6}{7}, -1 \rangle$

$\frac{3}{7}(x-3) + \frac{2}{7}(y-2) + \frac{6}{7}(z-6) - 1(w-7) = 0$

$\frac{3}{7}x - \frac{9}{7} + \frac{2}{7}y - \frac{4}{7} + \frac{6}{7}z - \frac{36}{7} + 7 = L(x,y,z) = \frac{3}{7}x + \frac{2}{7}y + \frac{6}{7}z + \frac{8}{7}$

or $\frac{3}{7}dx + \frac{2}{7}dy + \frac{6}{7}dz + 7$

$\frac{3}{7}(.02) + \frac{2}{7}(-.03) + \frac{6}{7}(-.01) = -.00857 = -3/350 \approx dz \quad \sqrt{3.02^2+1.97^2+5.99^2} \approx 6.991$

7. Use the chain rule to find $\frac{dz}{dt}$ for $z = \tan^{-1}\left(\frac{y}{x}\right)$, $x = e^t$, $y = 1 - e^{-t}$. (6 points)

$\frac{\partial z}{\partial x} = \frac{1}{1+\frac{y^2}{x^2}} \cdot \frac{-y}{x^2} = \frac{-y}{x^2+y^2} = \frac{e^{-t}-1}{e^{2t}+(1-e^{-t})^2} = \frac{e^{-t}-1}{e^{2t}+1-2e^{-t}+e^{-2t}}$

$\frac{\partial z}{\partial y} = \frac{1}{1+\frac{y^2}{x^2}} \cdot \frac{1}{x} \cdot \frac{x}{x} = \frac{x}{x^2+y^2} = \frac{e^t}{e^{2t}+1-2e^{-t}+e^{-2t}}$

$\frac{dx}{dt} = e^t$

$\frac{dy}{dt} = e^{-t}$

$\frac{dz}{dt} = \frac{e^{-t}-1}{e^{2t}+1-2e^{-t}+e^{-2t}} \cdot e^t + \frac{e^t}{e^{2t}+1-2e^{-t}+e^{-2t}} \cdot e^{-t}$

$= \frac{1-e^t+1}{e^{2t}+1-2e^{-t}+e^{-2t}} = \boxed{\frac{2-e^t}{e^{2t}+1-2e^{-t}+e^{-2t}}}$

8. Find $\frac{dy}{dx}$ for the implicit function $y\cos x = x^2 + y^2$. (5 points)

$F = x^2 + y^2 - y\cos x$

$F_x = 2x + y\sin x$ $\quad \frac{dy}{dx} = -\frac{F_x}{F_y} = -\frac{(2x + y\sin x)}{(2y - \cos x)} =$

$F_y = 2y - \cos x$

$$\boxed{\frac{2x + y\sin x}{\cos x - 2y}}$$

9. Find the directional derivative on the function $f(x, y, z) = xy + yz + xz$ at $P(1, -1, 3)$ in the direction of $Q(2,4,5)$. From the point P, in which direction is the rate of change a maximum? (7 points)

$\nabla f = \langle y+z,\ x+z,\ y+x \rangle \quad \nabla f(1,-1,3) = \langle 2, 4, 0 \rangle$

$\vec{PQ} = \langle 1, 5, 2 \rangle \quad \|\vec{PQ}\| = \sqrt{1+25+4} = \sqrt{30}$

$\hat{PQ} = \frac{1}{\sqrt{30}}\hat{i} + \frac{5}{\sqrt{30}}\hat{j} + \frac{2}{\sqrt{30}}\hat{k}$

$\|\nabla f\| = \sqrt{4+16+0} = \sqrt{20} = 2\sqrt{5}$

$\nabla f \cdot \hat{PQ} = \frac{2}{\sqrt{30}} + \frac{20}{\sqrt{30}} + 0 = \boxed{\frac{22}{\sqrt{30}}} = D_{\hat{PQ}}$

max @ $\langle \frac{1}{\sqrt{5}}, \frac{4}{\sqrt{5}}, 0 \rangle$

10. Sketch the gradient field of $f(x, y) = xy - 2x - 2y - x^2 - y^2$. Use the gradient field to sketch at least 5 level curves and locate any extrema. Based on the graph, determine if the extrema are maxima, minima, saddle points, or cannot be determined. Then use the second partials test on the same function to verify the results. (12 points)

$\langle \underset{f_x}{y-2-2x},\ \underset{f_y}{x-2-2y} \rangle$ $\quad f_{xx} = -2$

$f_{yy} = -2$

$f_{xy} = 1$

$y - 2 - 2x = 0 \quad x - 2 - 2y = 0$

$y = 2x + 2 \quad \frac{x-2}{2} = \frac{2y}{2}$

$2(2x + 2 = \frac{1}{2}x - 1) \quad \frac{1}{2}x - 1 = y$

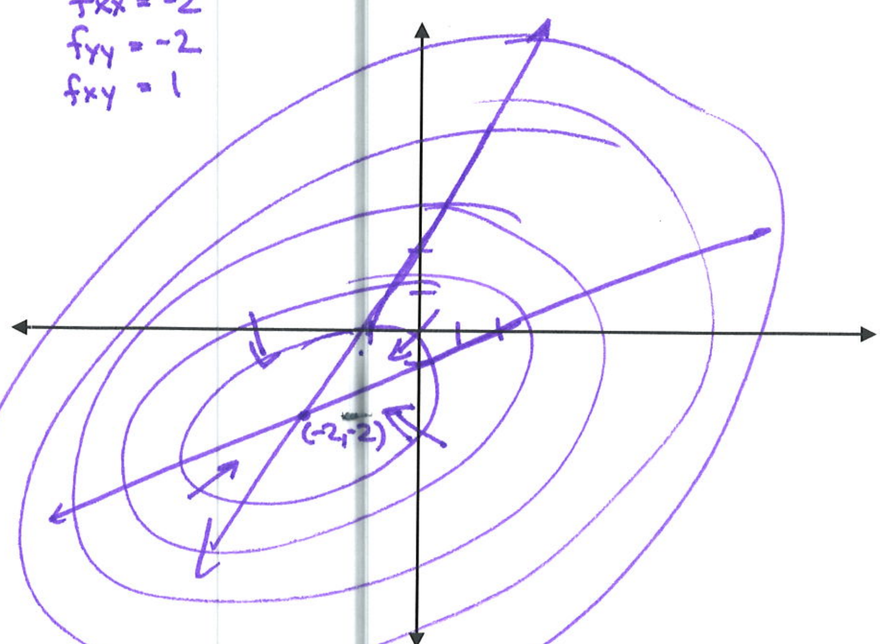
$4x + 4 = x - 2 \quad \frac{1}{2}(-2) - 1 = -2$

$-x - 4 \quad -x - 4$

$\frac{3x}{3} = \frac{-6}{3}$

$x = -2$

extrema at $(-2,-2)$ max



$\nabla f(0,0) = \langle -2, -2 \rangle$

$\nabla f(0,-2) = \langle -4, 2 \rangle$

$\nabla f(-2,0) = \langle 2, -4 \rangle$

$\nabla f(-3,-3) = \langle 1, 1 \rangle$

$D = (-2)(-2) - 1^2 = 4 - 1 = 3$ max or min

$f_{xx} < 0$ concave down $\Rightarrow$ max

11. Find the surface area of the hyperbolic paraboloid $z = x^2 - y^2$ that lies between $x^2 + y^2 = 1$ and $x^2 + y^2 = 4$. (8 points)

$F = x^2 - y^2 - z \quad \nabla F = \langle 2x, -2y, -1 \rangle \quad \|\nabla F\| = \sqrt{4x^2+4y^2+1} = \sqrt{4r^2+1}$

$\int_0^{2\pi}\int_1^2 \sqrt{4r^2+1}\, r\,dr\,d\theta$

$u = 4r^2+1$
$du = 8r\,dr$
$\frac{1}{8}du = r\,dr$

$\int u^{1/2} \cdot \frac{1}{8} du = \frac{1}{\cancel{8}4} \cdot \frac{\cancel{2}}{3} u^{3/2} = \frac{1}{12} u^{3/2}$

$\int_0^{2\pi} \frac{1}{12}(4r^2+1)^{3/2}\Big|_1^2 d\theta$

$= \frac{1}{12}\left[17^{3/2} - 5^{3/2}\right] \cdot 2\pi$

$\boxed{\frac{\pi}{6}\left[17^{3/2} - 5^{3/2}\right]}$

12. Find the Jacobian of the transformation $x = e^{-r}\cos\theta$, $y = e^{r}\sin\theta$. (6 points)

$\begin{vmatrix} -e^{-r}\cos\theta & -e^{-r}\sin\theta \\ e^{r}\sin\theta & e^{r}\cos\theta \end{vmatrix} = -\cos^2\theta + \sin^2\theta = \sin^2\theta - \cos^2\theta = -\cos 2\theta$

$\frac{\partial(x,y)}{\partial(r,\theta)} = \cos 2\theta$

13. Set up, but do not evaluate, $\iint_R y^2 dA$ for R bounded by $xy = 1, xy = 2, xy^2 = 1, xy^2 = 2$ using $u = xy, v = xy^2$. Complete the change of variables and sketch the region before the transformation. (8 points)

$u = xy$

$uy = v = xy \cdot y = xy^2$

$\boxed{y = \frac{v}{u}}$

$u = x \cdot \frac{v}{u}$

$\boxed{\frac{u^2}{v} = x}$

$y^2 = \frac{v^2}{u^2}$

$u \in [1,2]$
$v \in [1,2]$

$J = \begin{vmatrix} \frac{2u}{v} & -\frac{u^2}{v^2} \\ -\frac{v}{u^2} & \frac{1}{u} \end{vmatrix} = \frac{2\cancel{u}}{v} \cdot \frac{1}{\cancel{u}} - \frac{\cancel{u^2}}{v^{\cancel{2}}} \cdot \frac{\cancel{v}}{\cancel{u^2}} = \frac{2}{v} - \frac{1}{v} = \frac{1}{v}$

$\int_1^2\int_1^2 \frac{v^2}{u^2} \cdot \frac{1}{v}\, du\,dv$

$= \int_1^2\int_1^2 \frac{v}{u^2}\, du\,dv$

$\iint_R xy^2 dA =$

$\int_1^2\int_1^2 \frac{u^2}{v} \cdot \frac{v^2}{u^2} \cdot \frac{1}{v}\, du\,dv = \int_1^2\int_1^2 1\, du\,dv = (2-1)(2-1) = 1$

14. Set up the integral to find the surface area of $\vec{r}(u,v) = \cos^3 u \cos^3 v\,\hat{\imath} + \sin^3 u \cos^3 v\,\hat{\jmath} + \sin^3 v\,\hat{k}$ for $0 \le u \le \pi, 0 \le v \le 2\pi$. (8 points)

$$r_u = -3\cos^2 u \sin u \cos^3 v\,\hat{\imath} + 3\sin^2 u \cos u \cos^3 v\,\hat{\jmath} + 0\hat{k}$$

$$r_v = -3\cos^3 u \cos^2 v \sin v\,\hat{\imath} + -3\sin^3 u \cos^2 v \sin v\,\hat{\jmath} + 3\sin^2 v \cos v\,\hat{k}$$

$$\|r_u \times r_v\| = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ -3\cos^2 u \sin u \cos^3 v & 3\sin^2 u \cos u \cos^3 v & 0 \\ -3\cos^3 u \cos^2 v \sin v & -3\sin^3 u \cos^2 v \sin v & 3\sin^2 v \cos v \end{vmatrix} =$$

$$(9\sin^2 u \cos u \sin^2 v \cos^4 v)\hat{\imath} - (-9\cos^2 u \sin u \sin^2 v \cos^4 v)\hat{\jmath} +$$

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$$(\underbrace{9\sin^4 u \cos^2 u \cos^5 v \sin v + 9\cos^4 u \sin^2 u \cos^5 v \sin v}_{9\sin^4 u \cos^5 v \sin v\,\hat{k}})\hat{k}$$

$$\int_0^{\pi}\int_0^{2\pi} \sqrt{81\sin^4 u \cos^2 u \sin^4 v \cos^8 v + 81\cos^4 u \sin^2 u \sin^4 v \cos^8 v + 81\sin^8 u \cos^{10} v \sin^2 v}\, dv\, du$$

15. Find the surface integral $\iint_S x^2yz\,dS$ where S is the part $z = 1 + 2x + 3y$ above the rectangle $[0,3] \times [0,2]$. You don't need to evaluate the final integral. (7 points)

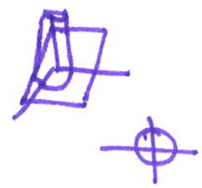
$$\nabla F = \langle 2, 3, -1 \rangle$$

$$\|\nabla F\| = \sqrt{4+9+1} = \sqrt{14}$$

$$\int_0^3\int_0^2 x^2y(1+2x+3y)\sqrt{14}\, dy\, dx$$

16. Use the divergence theorem to find the flux of $\vec{F}(x,y,z) = x^4\hat{\imath} - x^3z^2\hat{\jmath} + 4xy^2z\hat{k}$ where S is the surface bounded by cylinder $x^2 + y^2 = 1$ and $z = x + 2, z = 0$. (8 points)

$$\nabla \cdot F = 4x^3 - 0 + 4xy^2 = 4x^3 + 4xy^2$$



$$\int\int\int_0^{x+2}$$

$$\int_0^{2\pi}\int_0^1\int_0^{r\cos\theta+2} (4r^3\cos^3\theta + 4r^3\cos\theta\sin^2\theta)\, r\, dz\, dr\, d\theta$$

$$\int_0^{2\pi}\int_0^1 (4r^4\cos^3\theta + 4r^4\cos\theta\sin^2\theta)(r\cos\theta + 2)\, dr\, d\theta$$

$$\int_0^{2\pi}\int_0^1 4r^5\cos^4\theta + 4r^5\cos^2\theta\sin^2\theta + 8r^4\cos^3\theta + 8r^4\cos\theta\sin^2\theta\, dr\, d\theta$$

$$\int_0^{2\pi} \frac{2}{3}\cos^4\theta + \frac{2}{3}\cos^2\theta\sin^2\theta + \frac{8}{5}\cos^3\theta + \frac{8}{5}\cos\theta\sin^2\theta\, d\theta$$

$$\pi/2 + \frac{2}{3}\cdot\frac{\pi}{4} + 0 + 0 = \boxed{\frac{2\pi}{3}}$$

17. Find the absolute max/min of the function $f(x, y) = x^2 + y^2 + x^2y + 4$ over the region ~~|x|1 ≤~~ $|x| \le 1$, $|y| \leq 1$. (8 points)

$f_x = 2x + 2xy = 0 \qquad x = 0 \text{ or } y = -1$

$2x(1+y) = 0$

$f_y = 2y + x^2 = 0$

$x^2 = -2y$

$x = 0 \Rightarrow y = 0 \quad (0,0)$

$x^2 = 2 \qquad x = \sqrt{2} \quad (\sqrt{2}, -1)$

$f(x,1) = x^2 + 1 + x^2 + 4 = 2x^2 + 5$

$f' = 4x = 0 \qquad x = 0 \qquad (0,1)$

$f(x,-1) = x^2 + 1 - x^2 + 4 = 5 \quad f' = 0$

$f(1,y) = 1 + y^2 + y + 4 = y^2 + y + 5$

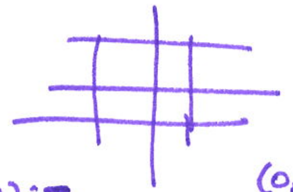
$f' = 2y + 1 = 0 \qquad y = -1/2$

$f(-1,y) = 1 + y^2 + y + 4$

$= y^2 + y + 5$

$f' = 2y + 1 = 0$

$y = -1/2$



MIN $\boxed{f(0,0) = 4}$

$f(\sqrt{2}, -1) = 5$

$\boxed{f(-\sqrt{2}, 1) = 9}$ MAX

$f(0,1) = 5$

$f(1, -1/2) = 19/4 = 4.75$

$f(-1, -1/2) = 19/4$

$f(1,1) = 7$

$f(-1,1) = 7$

$f(1,-1) = 5$

$f(-1,-1) = 5$

$(0,0)$
$(\sqrt{2}, -1)$
$(-\sqrt{2}, 1)$
$(0,1)$
$(1, -1/2)$
$(-1, -1/2)$
$(1,1)$ $(-1,1)$
$(1,-1)$ $(-1,-1)$

18. Find an equation of the tangent plane to the surface $\vec{r}(u, v) = (1 - u^2 - v^2)\hat{\imath} - v\hat{\jmath} - u\hat{k}$ at the point $(-1, -1, -1)$. (8 points)

$u = +1, \ v = 1$

$r_u = -2u\hat{\imath} + 0\hat{\jmath} - 1\hat{k}$

$r_v = -2v\hat{\imath} - \hat{\jmath} + 0\hat{k}$

$$\vec{r}_u \times \vec{r}_v = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ -2u & 0 & -1 \\ -2v & -1 & 0 \end{vmatrix} \quad (0-1)\hat{\imath} - (0-2v)\hat{\jmath} + (2u-0)\hat{k}$$

$$-\hat{\imath} + 2v\hat{\jmath} + 2u\hat{k} = \vec{r}_u \times \vec{r}_v \ (1,1) = \langle -1, 2, 2 \rangle$$

$$\boxed{-(x+1) + 2(y+1) + 2(z+1) = 0}$$

Useful formulas:

$$K = \frac{|f''(x)|}{\left(1 + [f'(x)]^2\right)^{\frac{3}{2}}}$$

$$K = \frac{\left\|\vec{r}'(t) \times \vec{r}''(t)\right\|}{\left\|\vec{r}'(t)\right\|^3}$$

14 cont'd

$$\int_0^{\pi}\int_0^{2\pi}\sqrt{\cancel{81}\sin^4u\cos^2u\,\sin^4v\,\cancel{\cos^8v}+\cancel{81}\cos^4u\,\cancel{\sin^2u}\,\sin^4v\,\cancel{\cos^8v}+\cancel{81}\sin^8u\cos^{10}v\,\cancel{\sin^2v}}\;dv\,du$$

$$\sin^2u\cos^2u\sin^2v \qquad \cos^4u \qquad \sin^2v \qquad \sin^6u\cos^2v$$

$$9\int_0^{\pi}\int_0^{2\pi}\sin u\,\sin v\,\cos^4v\sqrt{\sin^2u\cos^2u\sin^2v+\cos^4u\sin^2v+\sin^6u\cos^2v}\;dv\,du$$

$r(u,v) =$

14. $\cos u\cos v\,\hat{\imath}+\sin u\cos v\,\hat{\jmath}+\sin v\,\hat{k}$

$r_u = -\sin u\cos v\,\hat{\imath}+\cos u\cos v\,\hat{\jmath}+0\,\hat{k}$

$r_v = -\cos u\sin v\,\hat{\imath}-\sin u\sin v\,\hat{\jmath}+\cos v\,\hat{k}$

$$\begin{vmatrix}\hat{\imath} & \hat{\jmath} & \hat{k}\\ -\sin u\cos v & \cos u\cos v & 0\\ -\cos u\sin v & -\sin u\sin v & \cos v\end{vmatrix} = (\cos u\cos^2v-0)\hat{\imath}-(-\sin u\cos^2v-0)\hat{\jmath}$$

$$+(\sin^2u\cos v\sin v+\cos^2u\sin v\cos v)\hat{k}$$

$$=$$

$$\cos u\cos^2v\,\hat{\imath}+\sin u\cos^2v\,\hat{\jmath}+$$

$$\sin v\cos v\,\hat{k}$$

$$\|r_u\times r_v\| = \sqrt{\cos^2u\cos^4v+\sin^2u\cos^4v+\sin^2v\cos^2v} =$$

$$\sqrt{\cos^4v+\cos^2v\sin^2v} =$$

$$\sqrt{\cos^2v(\cancel{\cos^2v+\sin^2v})} = \cos v$$

$$\int_0^{2\pi}\int_0^{\pi}\cos v\;du\,dv$$