MAT 201, Quiz #14, Spring 2016 Name KEY

Instructions: Show all work. Answers without work required to obtain the solution will not receive full credit. Some questions may contain multiple parts: be sure to answer all of them. Give exact answers unless specifically asked to estimate.

1. The joint density function for a pair of random variables is given by $f(x,y) = Cx(1+y)$, $0 \le x \le 1,\ 0 \le y \le 2$ and is equal to zero elsewhere.
   a. Find C.

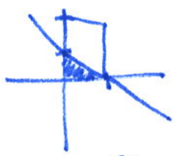
$$\int_0^1\int_0^2 Cx(1+y)\,dy\,dx = C\int_0^1 x\left(y+\tfrac{1}{2}y^2\right)\Big|_0^2 dx = C\int_0^1 x(2+2)\,dx =$$

$$4C\int_0^1 x\,dx = 4C\cdot\tfrac{1}{2}x^2\Big|_0^1 = 4C\cdot\tfrac{1}{2} = 2C \Rightarrow \text{must} = 1 \text{ so}$$

$$2C = 1 \Rightarrow C = \tfrac{1}{2}$$

   b. Find $P(X+Y \le 1)$.

$y = -x+1$



$$\tfrac{1}{2}\int_0^1\int_0^{-x+1} x(1+y)\,dy\,dx = \tfrac{1}{2}\int_0^1 x\left(y+\tfrac{1}{2}y^2\right)\Big|_0^{-x+1} dx$$

$$= \tfrac{1}{2}\int_0^1 x\left(-x+1+\tfrac{1}{2}(\underbrace{-x+1}_{1-x})^2\right)dx = \tfrac{1}{2}\int_0^1 x\left(1-x+\tfrac{1}{2}(1-2x+x^2)\right)dx$$

$$= \tfrac{1}{2}\int_0^1 x\left(1-x+\tfrac{1}{2}-x+\tfrac{1}{2}x^2\right)dx = \tfrac{1}{2}\int_0^1 x\left(\tfrac{3}{2}-2x+\tfrac{1}{2}x^2\right)dx = \tfrac{1}{2}\int_0^1 \tfrac{3}{2}x-2x^2+\tfrac{1}{2}x^3\,dx$$

$$= \tfrac{1}{2}\left[\tfrac{3}{4}x^2-\tfrac{2}{3}x^3+\tfrac{1}{8}x^4\Big|_0^1 = \tfrac{1}{2}\left(\tfrac{3}{4}-\tfrac{2}{3}+\tfrac{1}{8}\right) = \boxed{\tfrac{5}{48}} \approx .104 \text{ or } 10.4\%$$

2. Verify that the area of the surface of a sphere with radius R is $A = 4\pi R^2$.

$x^2+y^2+z^2 = R^2 \qquad z = \sqrt{R^2-x^2-y^2}$

$$\nabla z = \tfrac{1}{2}(R^2-x^2-y^2)^{-1/2}(-2x)\hat{i} + \tfrac{1}{2}(R^2-x^2-y^2)^{-1/2}(-2y)\hat{j} - \hat{k}$$

$$= \frac{-x}{\sqrt{R^2-x^2-y^2}}\hat{i} + \frac{-y}{\sqrt{R^2-x^2-y^2}}\hat{j} - \hat{k} \qquad \|\nabla z\| = \sqrt{1+\frac{x^2+y^2}{R^2-x^2-y^2}} =$$

$$\sqrt{\frac{R^2-x^2-y^2+x^2+y^2}{R^2-x^2-y^2}} = \sqrt{\frac{R^2}{R^2-x^2-y^2}}$$

$$\int_0^{2\pi}\int_0^R R(R^2-r^2)^{-1/2}\,r\,dr\,d\theta =$$

$$-R\int_0^{2\pi}(R^2-r^2)^{1/2}\Big|_0^R d\theta = -R\int_0^{2\pi} 0-\underbrace{\sqrt{R^2}}_{=R}\,d\theta =$$

$$+R^2\int_0^{2\pi} d\theta = 2\pi R^2 \leftarrow \text{top half of sphere}$$

$u = R^2-r^2$

$du = -2r\,dr$

$-\tfrac{1}{2}du = r\,dr$

$\int -\tfrac{1}{2}u^{-1/2}\,du$

$-\tfrac{1}{2}\cdot 2u^{1/2}$

$$2\times 2\pi R^2 = \boxed{4\pi R^2}$$

3. Find the center of mass for the tetrahedron bounded by $x = 0, y = 0, z = 0, x+y+z = 1$ with density $\rho(x,y,z) = y$.

$z = -x-y+1$

$y = -x+1$

$$M = \int_0^1\int_0^{-x+1}\int_0^{-x-y+1} y\,dz\,dy\,dx = \int_0^1\int_0^{-x+1} -xy-y^2+y\,dy\,dx = \int_0^1 -\tfrac{1}{2}xy^2-\tfrac{1}{3}y^3+\tfrac{1}{2}y^2\Big|_0^{-x+1}$$

$$\int_0^1 -\tfrac{1}{2}x(1-x)^2-\tfrac{1}{3}(1-x^3)+\tfrac{1}{2}(1-x)^2\,dx = \int_0^1 -\tfrac{1}{2}x(1-2x+x^2)-\tfrac{1}{3}(1-3x+3x^2-x^3)+\tfrac{1}{2}(1-2x+x^2)\,dx$$

$$= \int_0^1 -\tfrac{1}{2}x+x^2-\tfrac{1}{2}x^3-\tfrac{1}{3}+\cancel{x}-\cancel{x^2}+\tfrac{1}{3}x^3+\tfrac{1}{2}-\cancel{x}+\tfrac{1}{2}x^2\,dx = \int_0^1 \tfrac{1}{6}-\tfrac{1}{2}x+\tfrac{1}{2}x^2-\tfrac{1}{6}x^3\,dx =$$

$$\tfrac{1}{6}x-\tfrac{1}{4}x^2+\tfrac{1}{6}x^3-\tfrac{1}{24}x^4\Big|_0^1 = \tfrac{1}{6}-\tfrac{1}{4}+\tfrac{1}{6}-\tfrac{1}{24} = \boxed{\tfrac{1}{24}} \text{ mass}$$

$M_{yz} = \int_0^1\int_0^{-x+1}\int_0^{-x-y+1} xy\,dz\,dy\,dx = \int_0^1\int_0^{-x+1} xy(-x-y+1)\,dy\,dx =$

$\int_0^1\int_0^{-x+1} -x^2y - xy^2 + xy\,dy\,dx = \int_0^1 -\frac{1}{2}x^2y^2 - \frac{1}{3}xy^3 + \frac{1}{2}xy^2\Big|_0^{-x+1} =$

$\int_0^1 -\frac{1}{2}x^2(1-x)^2 - \frac{1}{3}x(1-x)^3 + \frac{1}{2}x(1-x)^2\,dx = \int_0^1 -\frac{1}{2}x^2(1-2x+x^2) - \frac{1}{3}x(1-3x+3x^2-x^3) +$
$\frac{1}{2}x(1-2x+x^2)\,dx =$

$\int_0^1 -\frac{1}{2}x^2 + \cancel{x^3} - \frac{1}{2}x^4 - \frac{1}{3}x + \cancel{x^2} - \cancel{x^3} + \frac{1}{3}x^4 + \frac{1}{2}x - \cancel{x^2} + \frac{1}{2}x^3\,dx =$

$\int_0^1 \frac{1}{6}x - \frac{1}{2}x^2 + \frac{1}{2}x^3 - \frac{1}{6}x^4\,dx = \frac{1}{12}x^2 - \frac{1}{6}x^3 + \frac{1}{8}x^4 - \frac{1}{30}x^5\Big|_0^1 =$

$\frac{1}{12} - \frac{1}{6} + \frac{1}{8} - \frac{1}{30} = \boxed{\frac{1}{120}}$

$\bar{x} = \frac{\frac{1}{120}}{\frac{1}{24}} = \frac{24}{120} = \boxed{\frac{1}{5}}$

$M_{xz} = \int_0^1\int_0^{-x+1}\int_0^{x-y+1} y^2\,dz\,dy\,dx = \int_0^1\int_0^{1-x} y^2(-x-y+1)\,dy\,dx = \int_0^1\int_0^{1-x} -xy^2 - y^3 + y^2\,dy\,dx =$

$\int_0^1 -\frac{1}{3}xy^3 - \frac{1}{4}y^4 + \frac{1}{3}y^3\Big|_0^{1-x} = \int_0^1 -\frac{1}{3}x(1-x)^3 - \frac{1}{4}(1-x)^4 + \frac{1}{3}(1-x)^3\,dx =$

$\int_0^1 -\frac{1}{3}x(1-3x+3x^2-x^3) - \frac{1}{4}(1-4x+6x^2-4x^3+x^4) + \frac{1}{3}(1-3x+3x^2-x^3)\,dx =$

$\int_0^1 -\frac{1}{3}x + x^2 - \cancel{x^3} + \frac{1}{3}x^4 - \frac{1}{4} + \cancel{x} - \frac{3}{2}x^2 + \cancel{x^3} - \frac{1}{4}x^4 + \frac{1}{3} - \cancel{x} + x^2 - \frac{1}{3}x^3\,dx =$

$\int_0^1 \frac{1}{12} - \frac{1}{3}x + \frac{1}{2}x^2 - \frac{1}{3}x^3 + \frac{1}{12}x^4\,dx = \frac{1}{12}x - \frac{1}{6}x^2 + \frac{1}{6}x^3 - \frac{1}{12}x^4 + \frac{1}{60}x^5\Big|_0^1 = \cancel{\frac{1}{12}} - \cancel{\frac{1}{6}} + \cancel{\frac{1}{6}} - \cancel{\frac{1}{12}} + \frac{1}{60} = \boxed{\frac{1}{60}}$

$\bar{y} = \frac{\frac{1}{60}}{\frac{1}{24}} = \frac{24}{60} = \boxed{\frac{2}{5}}$

$M_{xy} = \int_0^1\int_0^{1-x}\int_0^{1-x-y} zy\,dz\,dy\,dx = \int_0^1\int_0^{1-x} \frac{1}{2}z^2y\Big|_0^{1-x-y} dy\,dx = \int_0^1\int_0^{1-x} \frac{1}{2}(1-x-y)^2y\,dy\,dx =$

$\int_0^1\int_0^{1-x} \frac{1}{2}y(1-2x-2y+2xy+x^2+y^2)\,dy\,dx =$

$(1-x-y)(1-x-y) = 1-x-y-x$
$+x^2+xy-y+xy+y^2$

$\int_0^1\int_0^{1-x} \frac{1}{2}y - xy - y^2 + xy^2 + \frac{1}{2}x^2y + \frac{1}{2}y^3\,dy\,dx = \int_0^1 \frac{1}{4}y^2 - \frac{1}{2}xy^2 - \frac{1}{3}y^3 + \frac{1}{3}xy^3 + \frac{1}{4}x^2y^2 + \frac{1}{8}y^4\Big|_0^{1-x}$

$\int_0^1 \frac{1}{4}(1-x)^2 - \frac{1}{2}x(1-x)^2 - \frac{1}{3}(1-x)^3 + \frac{1}{3}x(1-x)^3 + \frac{1}{4}x^2(1-x)^2 + \frac{1}{8}(1-x)^4\,dx =$

$\int_0^1 \frac{1}{4}(1-2x+x^2) - \frac{1}{2}x(1-2x+x^2) - \frac{1}{3}(1-3x+3x^2-x^3) + \frac{1}{3}x(1-3x+3x^2-x^3) + \frac{1}{4}x^2(1-2x+x^2) + \frac{1}{8}(1-4x+6x^2-4x^3+x^4)\,dx$

$= \int_0^1 \frac{1}{4} - \cancel{\frac{1}{2}x} + \frac{1}{4}x^2 - \cancel{\frac{1}{2}x} + \cancel{x^2} - \cancel{\frac{1}{2}x^3} - \frac{1}{3} + \cancel{x} - \cancel{x^2} + \frac{1}{3}x^3 + \frac{1}{3}x - x^2 + \cancel{x^3} - \frac{1}{3}x^4 + \frac{1}{4}x^2 - \cancel{\frac{1}{2}x^3} + \frac{1}{4}x^4 + \frac{1}{8} - \frac{1}{2}x + \frac{3}{4}x^2 - \frac{1}{2}x^3 + \frac{1}{8}x^4\,dx$

$= \int \frac{1}{24} - \frac{1}{6}x + \frac{1}{4}x^2 - \frac{1}{6}x^3 + \frac{1}{24}x^4\,dx = \frac{1}{24}x - \frac{1}{12}x^2 + \frac{1}{12}x^3 - \frac{1}{24}x^4 + \frac{1}{120}x^5\Big|_0^1 = \frac{1}{24} - \cancel{\frac{1}{12}} + \cancel{\frac{1}{12}} - \frac{1}{24} + \frac{1}{120} = \frac{1}{120}$ $\quad \bar{z} = \frac{\frac{1}{120}}{\frac{1}{24}} = \frac{24}{120} = \boxed{\frac{1}{5}}$

Center of mass $= \left(\frac{1}{5}, \frac{2}{5}, \frac{1}{5}\right)$