MAT 201, Quiz #6, Spring 2016 Name KEY

Instructions: Show all work. Answers without work required to obtain the solution will not receive full credit. Some questions may contain multiple parts: be sure to answer all of them. Give exact answers unless specifically asked to estimate.

1. Evaluate the following for $f(x,y,z) = x^2yz - xyz^3$ and $\vec{F}(x,y,z) = y^2z^3\hat{\imath} + 2xyz^3\hat{\jmath} + 3xy^2z^2\hat{k}$
   a. $\vec{\nabla} f$

$$\langle 2xyz - yz^3,\ x^2z - xz^3,\ x^2y - 3xyz^2 \rangle$$

   b. $\vec{\nabla} \times \vec{F}$

$$\begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ y^2z^3 & 2xyz^3 & 3xy^2z^2 \end{vmatrix} = (6xyz^2 - 6xyz^2)\hat{\imath} - (3y^2z^2 - 3y^2z^2)\hat{\jmath} + (2yz^3 - 2yz^3)\hat{k}$$

$$= \vec{0}$$

   c. $\vec{\nabla} \cdot \vec{F}$

$$0 + 2xz^3 + 6xy^2z$$

$$= 2xz^3 + 6xy^2z$$

   d. $\nabla^2 f$

$$2yz + 0 + (-6xyz) = 2yz - 6xyz$$

2. Integrate.
   a. $\int_0^1 \int_0^3 e^{x+3y}\,dx\,dy$ $= \int_0^1 e^{3y} \int_0^3 e^x\,dx\,dy = \int_0^1 e^{3y} \left(e^x \Big|_0^3 \right) dy =$

$$\int_0^1 e^{3y}(e^3 - 1)\,dy = (e^3-1)\tfrac{1}{3}e^{3y}\Big|_0^1 = \tfrac{1}{3}(e^3-1)(e^3-1) =$$

$$\tfrac{1}{3}(e^3-1)^2$$

   b. $\iint_R \frac{1}{1+x+y}\,dA$ over the region $1 \le x \le 3, 1 \le y \le 2$

$$\int_1^3 \int_1^2 \frac{1}{1+x+y}\,dy\,dx = \int_1^3 \ln(1+x+y)\Big|_1^2 dx = \int_1^3 (\ln(x+3) - \ln(x+2))\,dx$$

$u = \ln(x+3) - \ln(x+2)$ $\quad dv = dx$

$du = \frac{1}{x+3} - \frac{1}{x+2}\,dx$ $\quad v = x$

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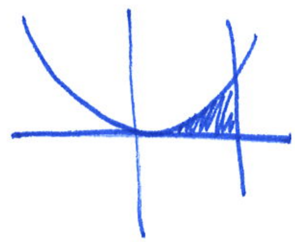
c. $\iint_D x\cos(y)\,dA$ over the region D bounded by $y = 0, y = x^2, x = 1$. Sketch the graph of the region.

$$\int_0^1 \int_0^{x^2} x\cos y \, dy\, dx =$$



$$\int_0^1 x \sin y \Big|_0^{x^2} dx = \int_0^1 x \sin(x^2)\, dx =$$

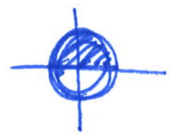
$$-\frac{1}{2}\cos(x^2)\Big|_0^1 = -\frac{1}{2}(\cos 1 - 1)$$

$$= \frac{1}{2}(1-\cos 1)$$

$u = x^2$
$du = 2x\,dx$
$x\,dx = \frac{1}{2}du$

d. $\iint_D x^2y\,dA$ where D is the top half of a disk centered at the origin with radius 5.

$x = r\cos\theta \quad y = r\sin\theta$
$x^2y = r^3\cos^2\theta\sin\theta \quad dA = r\,dr\,d\theta$



$$\int_0^\pi \int_0^5 r^4 \cos^2\theta \sin\theta\, dr\, d\theta = \int_0^\pi \frac{1}{5} r^5 \Big|_0^5 \cos^2\theta \sin\theta\, d\theta =$$

$$625\int_0^\pi \cos^2\theta\sin\theta\, d\theta = -625 \cdot \frac{1}{3}\cos^3\theta \Big|_0^\pi = -\frac{625}{3}(-1-1) = \boxed{\frac{1250}{3}}$$

$u = \cos\theta$
$-du = \sin\theta\, d\theta$
$-\int u^2\, du$

3. Find $\int_C \vec{F}\cdot d\vec{r}$ for $\vec{F}(x,y) = xy\hat{\imath} + 3y^2\hat{\jmath}$ over the curve $\vec{r}(t) = 11t^4\hat{\imath} + t^3\hat{\jmath},\ 0 \le t \le 1$

$$\vec{F}(t) = 11t^4 \cdot t^3\,\hat{\imath} + 3(t^3)^2\,\hat{\jmath} \qquad r'(t) = 44t^3\,\hat{\imath} + 3t^2\,\hat{\jmath}$$

$$= 11t^7\,\hat{\imath} + 3t^6\,\hat{\jmath}$$

$$\vec{F}\cdot d\vec{r} = (484t^{10} + 9t^8)\,dt$$

$$\int_0^1 484t^{10} + 9t^8\, dt = \frac{484}{11}t^{11} + \frac{9}{9}t^9 \Big|_0^1$$

$$44(1)^{11} + (1)^9 - 0 = \boxed{45}$$

$$x\left[\ln(x+3)-\ln(x+2)\right]-\int_1^3 \frac{x}{x+3}-\frac{x}{x+2}\,dx$$

$$\begin{array}{r} 1-\frac{3}{x+3} \\ x+3\,\big)\overline{\;x\;} \\ \underline{-x-3} \\ -3 \end{array} \qquad \begin{array}{r} 1-\frac{2}{x+2} \\ x+2\,\big)\overline{\;x\;} \\ \underline{-x-2} \\ -2 \end{array}$$

$$x\ln\left(\frac{x+3}{x+2}\right)-\int_1^3 1-\frac{3}{x+3}-\left(1-\frac{2}{x+2}\right)dx =$$

$$x\ln\left(\frac{x+3}{x+2}\right)-\int_1^3 \frac{2}{x+2}-\frac{3}{x+3}\,dx =$$

$$x\ln\left(\frac{x+3}{x+2}\right)-\left[2\ln(x+2)-3\ln(x+3)\right]\Big|_1^3 =$$

$$x\ln\left(\frac{x+3}{x+2}\right)+\ln\left[\frac{(x+3)^3}{(x+2)^2}\right]\Bigg|_1^3 = 3\ln\left(\frac{6}{5}\right)+\ln\left(\frac{6^3}{5^2}\right)-1\cdot\ln\left(\frac{4}{3}\right)-\ln\left(\frac{4^3}{3^2}\right)$$

$$= \ln\left(\frac{6^3}{5^3}\right)+\ln\left(\frac{6^3}{5^2}\right)-\ln\left(\frac{4}{3}\right)-\ln\left(\frac{4^3}{3^2}\right)$$

$$= \ln(6^3)-\ln 5^3+\ln 6^3-\ln 5^2-\ln 4+\ln 3-\ln 4^3+\ln 3^2$$

$$= 2\ln 6^3-3\ln 5-2\ln 5-\ln 4-3\ln 4+\ln 3+2\ln 3$$

$$= \ln 6^6-5\ln 5-4\ln 4+3\ln 3$$

$$= \ln 6^6-\ln 5^5-\ln 4^4+\ln 3^3 = \ln\left(\frac{6^6 3^3}{5^5 4^4}\right) = \ln\left(\frac{1259712}{800{,}000}\right)$$

$$\text{or } \ln\left[\left(\frac{6^6}{4^4}\right)\left(\frac{3^3}{5^5}\right)\right] = \ln\left[\left(\frac{729}{4}\right)\left(\frac{3^3}{5^5}\right)\right] = \ln\left(\frac{19683}{12500}\right)$$