MAT 201, Quiz #7, Spring 2016 Name KEY

**Instructions**: Show all work. Answers without work required to obtain the solution will not receive full credit. Some questions may contain multiple parts: be sure to answer all of them. Give exact answers unless specifically asked to estimate.

1. Evaluate each integral.

   a. $\iiint_E y\,dV$ where $E$ is the region defined by $0 \le x \le 3, 0 \le y \le x, x-y \le z \le x+y$

$$\int_0^3\int_0^x\int_{x-y}^{x+y} y\,dz\,dy\,dx = \int_0^3\int_0^x yz\Big|_{x-y}^{x+y} dy\,dx = \int_0^3\int_0^x y\,(x+y-x+y)\,dy\,dx =$$

$$\int_0^3\int_0^x 2y^2\,dy\,dx = \int_0^3 \tfrac{2}{3}y^3\Big|_0^x dx = \int_0^3 \tfrac{2}{3}x^3\,dx = \tfrac{1}{6}x^4\Big|_0^3 = \tfrac{1}{6}\cdot 81 = \boxed{\tfrac{27}{2}}$$

   b. $\iiint_E x\,dV$ where $E$ is bounded by $x = 4y^2 + 4z^2, x = 4$

$\iiint z\,dV$   $x \leftrightarrow z$   $\Rightarrow$   $z = 4x^2+4y^2$   $z = 4$

$z = 4r^2$   $4 = 4r^2$   $r^2 = 1$

$$\int_0^{2\pi}\int_0^1\int_{4r^2}^4 zr\,dz\,dr\,d\theta = \int_0^{2\pi}\int_0^1 \tfrac{1}{2}z^2\Big|_{4r^2}^4 r\,dr\,d\theta =$$

$$\int_0^{2\pi}\int_0^1 \tfrac{1}{2}(16-16r^4)r\,dr\,d\theta = \int_0^{2\pi}\int_0^1 8r-8r^5\,dr\,d\theta = \int_0^{2\pi} 4r^2-\tfrac{4}{3}r^6\Big|_0^1 d\theta =$$

$$\int_0^{2\pi} 4-\tfrac{4}{3}\,d\theta = \int_0^{2\pi}\tfrac{8}{3}\,d\theta = \tfrac{8}{3}\theta\Big|_0^{2\pi} = \boxed{\tfrac{16\pi}{3}}$$

$\tfrac{12}{3}-\tfrac{4}{3}=\tfrac{8}{3}$

2. Convert the integral to the indicated coordinate system and evaluate it.

   a. $\int_{-3}^{3}\int_0^{\sqrt{9-x^2}}\int_0^{9-x^2-y^2}\sqrt{x^2+y^2}\,dz\,dy\,dx$ (cylindrical)   $r=3$

$z = 9-x^2-y^2 \Rightarrow z = 9-r^2$

$\sqrt{x^2+y^2} = \sqrt{r^2} = r$

$$\int_0^{\pi}\int_0^3\int_0^{9-r^2} r^2\,dz\,dr\,d\theta = \int_0^{\pi}\int_0^3 9r^2-r^4\,dr\,d\theta =$$

$$\int_0^{\pi} 3r^3-\tfrac{1}{5}r^5\Big|_0^3 d\theta = \int_0^{\pi} 81-\tfrac{243}{5}\,d\theta = \int_0^{\pi}\tfrac{162}{5}\,d\theta$$

$$\boxed{\tfrac{162}{5}\pi}$$

   b. $\int_0^1\int_0^{\sqrt{1-x^2}}\int_{\sqrt{x^2+y^2}}^{\sqrt{2-x^2-y^2}} xy\,dz\,dy\,dx$ (spherical)

$x^2+y^2+z^2=2$   $\rho=\sqrt{2}$

$\varphi=\pi/4$


$\boxed{\tfrac{4\sqrt{2}}{15}-\tfrac{1}{3}}$

intersects at $2x^2+2y^2=2$

$x^2+y^2=1$

$$\int_0^{\pi/2}\int_0^{\pi/4}\int_0^{\sqrt{2}} \rho^4\sin^3\varphi\cos\theta\sin\theta\,d\rho\,d\varphi\,d\theta =$$

$$\int_0^{\pi/2}\int_0^{\pi/4}\tfrac{1}{5}\rho^5\sin^3\varphi\cos\theta\sin\theta\Big|_0^{\sqrt{2}} d\varphi\,d\theta = \tfrac{4\sqrt{2}}{5}\int_0^{\pi/2}\underbrace{\cos\theta\sin\theta\,d\theta}_{\frac{1}{2}\sin^2\theta}\cdot\int_0^{\pi/4}\sin^3\varphi\,d\varphi$$


$$\tfrac{4\sqrt{2}}{5}\left[\tfrac{1}{2}\sin^2\theta\Big|_0^{\pi/2}\cdot\left[-\cos\varphi+\tfrac{1}{3}\cos^3\varphi\Big|_0^{\pi/4}\right.\right. =$$

$$\tfrac{\cancel{4}^{2}\sqrt{2}}{5}\tfrac{1}{\cancel{2}}\cdot\left[-\tfrac{1}{\sqrt{2}}+1+\tfrac{1}{3}\left(\tfrac{1}{\sqrt{2}}\right)^3-\tfrac{1}{3}(1)^3\right] = \tfrac{2\sqrt{2}}{5}\left[\tfrac{2}{3}-\tfrac{6}{6\sqrt{2}}+\tfrac{1}{6\sqrt{2}}\right] =$$

$$\tfrac{2\sqrt{2}}{5}\left[\tfrac{2}{3}-\tfrac{5}{6\sqrt{2}}\right] = \boxed{\tfrac{4\sqrt{2}}{15}-\tfrac{1}{3}}$$

$\rho\sin\varphi\cos\theta\,\rho\sin\varphi\sin\theta\,\rho\sin\varphi\,d\rho\,d\varphi\,d\theta$

$\sin^2\varphi\sin\varphi = (1-\cos^2\varphi)\sin\varphi$

$\sin\varphi - \cos^2\varphi\sin\varphi$

3. Find the potential function for $\vec{F}(x,y,z) = yz\hat{\imath} + xz\hat{\jmath} + (xy + 2z)\hat{k}$ and use that to evaluate the line integral $\int_C \vec{F} \cdot d\vec{r}$ over the path from $(1,0,-2)$ to $(4,6,3)$.

$\int yz\,dx = xyz + g(y,z)$

$\int xz\,dy = xyz + h(x,z)$

$\int xy + 2z = xyz + z^2 = i(x,y)$

$f(x,y,z) = xyz + z^2 + k$

$f(4,6,3) - f(1,0,-2) =$

$4 \cdot 6 \cdot 3 + 9 - (0 + 4) = 81 - 4 = \boxed{77}$

4. Use Green's Theorem to evaluate $\oint_C xydx + x^2dy$ over the rectangle with vertices $(0,0), (3,0), (3,1), (0,1)$.

$\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} = 2x - x = x$

$\int_0^3 \int_0^1 x\,dy\,dx = \int_0^3 xy \Big|_0^1 dx =$

$\int_0^3 x\,dx = \frac{1}{2}x^2 \Big|_0^3 = \boxed{\frac{9}{2}}$