MAT 212, Final Exam, Spring 2016 Name KEY

Instructions: Show all work. Answers without work required to obtain the solution will not receive full credit. Some questions may contain multiple parts: be sure to answer all of them. Give exact answers unless specifically asked to estimate.

1. Rewrite $u^{IV} - u = 0$ as a system of first order linear equations. (6 points)

$u = x_1 \quad x_1' = x_2 = u' \quad x_2' = x_3 = u'' \quad x_3' = x_4 = u''' \quad x_4' = u^{IV}$

$x_4' = x_1$

$$\rightarrow \boxed{\begin{aligned} x_1' &= x_2 \\ x_2' &= x_3 \\ x_3' &= x_4 \\ x_4' &= x_1 \end{aligned}}$$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}' = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}$$

2. Rewrite $\begin{cases} x_1' = x_1 - 2x_2 \\ x_2' = 3x_1 - 4x_2 \end{cases}$ as a single second order equation. (6 points)

$\dfrac{x_1' - x_1}{-2} = \dfrac{-2x_2}{-2} \Rightarrow x_2 = -\frac{1}{2}x_1' + \frac{1}{2}x_1$

$x_2' = \left(-\frac{1}{2}x_1' + \frac{1}{2}x_1\right)' = 3x_1 - 4x_2 = 3x_1 - 4\left(-\frac{1}{2}x_1' + \frac{1}{2}x_1\right)$

$\left(= -\frac{1}{2}x_1'' + \frac{1}{2}x_1' = 3x_1 + 2x_1' - 2x_1\right) \cdot -2$

$x_1'' - x_1' = -6x_1 - 4x_1' + 4x_1$

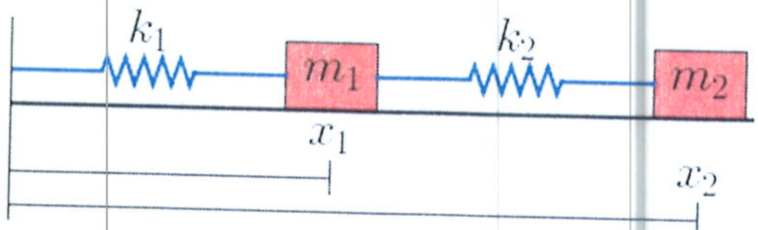
$x_1'' - x_1' = -2x_1 - 4x_1'$

$\underline{\quad +4x_1' \qquad\qquad +4x_1'}$

$\rightarrow x_1'' + 3x_1' + 2x_1 = 0$

$$\boxed{u'' + 3u' + 2u = 0}$$

3. Use the image to construct a system of second order equations to solve for the location of the masses at any time t given that $k_1 = 4, k_2 = 2, m_1 = 5, m_2 = 20$. You do not need to solve, just set it up. (10 points)



$m_1 x_1'' + (k_1 + k_2)x_1 - k_2 x_2 = 0 \qquad 5x_1'' + 6x_1 - 2x_2 = 0$

$m_2 x_2'' + k_2 x_2 - k_2 x_1 = 0 \qquad 20x_2'' + 2x_2 - 2x_1 = 0$

$x_3 = x_1'$

$x_4 = x_2'$

$x_3' = x_1''$

$x_4' = x_2''$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}' = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -6/5 & 2/5 & 0 & 0 \\ 1/10 & -1/10 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}$$

4. Prove that $\vec{x} = \begin{pmatrix} 4 \\ 2 \end{pmatrix} e^{2t}$ satisfies the differential equation $\vec{x}' = \begin{pmatrix} 3 & -2 \\ 2 & -2 \end{pmatrix} \vec{x}$. (6 points)

$$x' = \begin{pmatrix} 4 \\ 2 \end{pmatrix} 2e^{2t} = \begin{pmatrix} 8 \\ 4 \end{pmatrix} e^{2t}$$

$$\begin{pmatrix} 3 & -2 \\ 2 & -2 \end{pmatrix} \begin{pmatrix} 4 \\ 2 \end{pmatrix} e^{2t} = \begin{pmatrix} 12-4 = 8 \\ 8-4 = 4 \end{pmatrix} e^{2t} \quad \checkmark$$

5. Solve the system $\vec{x} = \begin{pmatrix} 1 & 1 \\ 4 & -2 \end{pmatrix} \vec{x}$. Sketch the direction field and several sample trajectories. (14 points)

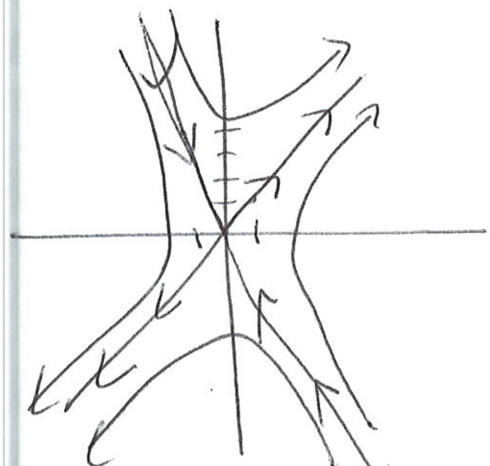
$$(1-\lambda)(-2-\lambda) - 4 = 0 \qquad \vec{x} = c_1 \begin{pmatrix} -1 \\ 4 \end{pmatrix} e^{-3t} + c_2 \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{t}$$

$$\lambda^2 + \lambda - 2 - 4 = 0$$

$$\lambda^2 + \lambda - 6 = 0$$

$$(\lambda+3)(\lambda-2) = 0$$

$$\lambda = -3, \ \lambda = +2$$



$$\begin{pmatrix} 1+3 & 1 \\ 4 & -2+3 \end{pmatrix} = \begin{pmatrix} 4 & 1 \\ 4 & 1 \end{pmatrix} \quad \begin{matrix} 4x_1 = -x_2 \\ x_1 = -\frac{1}{4}x_2 \\ x_2 = x_2 \end{matrix} \quad \begin{pmatrix} -1 \\ 4 \end{pmatrix}$$

$$\begin{pmatrix} 1-2 & 1 \\ 4 & -2-2 \end{pmatrix} = \begin{pmatrix} -1 & 1 \\ 4 & -4 \end{pmatrix} \quad \begin{matrix} x_1 = x_2 \\ x_2 = x_2 \end{matrix} \quad \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

6. Use the information from the previous problem to solve $\vec{x}' = \begin{pmatrix} 1 & 1 \\ 4 & -2 \end{pmatrix} \vec{x} + \begin{pmatrix} e^{-2t} \\ -2e^{t} \end{pmatrix}$. (8 points)

$$\Psi = \begin{pmatrix} -e^{-3t} & e^{2t} \\ 4e^{-3t} & e^{2t} \end{pmatrix} \qquad \Psi^{-1} = \frac{1}{\underset{-5e^{-t}}{-e^{-t} - 4e^{-t}}} \begin{pmatrix} e^{2t} & -e^{2t} \\ -4e^{-3t} & -e^{-3t} \end{pmatrix} = \begin{pmatrix} -1/5\,e^{3t} & 1/5\,e^{3t} \\ 4/5\,e^{-2t} & 1/5\,e^{-2t} \end{pmatrix}$$

$$\begin{pmatrix} -1/5\,e^{3t} & 1/5\,e^{3t} \\ 4/5\,e^{-2t} & 1/5\,e^{-2t} \end{pmatrix} \begin{pmatrix} e^{-2t} \\ -2e^{t} \end{pmatrix} = \begin{pmatrix} -1/5\,e^{t} - 2/5\,e^{4t} \\ 4/5\,e^{-4t} - 2/5\,e^{-t} \end{pmatrix}$$

$$\int \begin{pmatrix} -1/5\,e^{t} - 2/5\,e^{4t} \\ 4/5\,e^{-4t} - 2/5\,e^{-t} \end{pmatrix} dt = \begin{pmatrix} -1/5\,e^{t} - \frac{1}{10}e^{4t} \\ -1/5\,e^{-4t} + 2/5\,e^{-t} \end{pmatrix} \qquad \begin{pmatrix} -e^{-3t} & e^{2t} \\ 4e^{-3t} & e^{2t} \end{pmatrix} \begin{pmatrix} -1/5\,e^{t} - \frac{1}{10}e^{4t} \\ -1/5\,e^{-4t} + 2/5\,e^{-t} \end{pmatrix} =$$

$$\begin{pmatrix} \cancel{1/5\,e^{-2t}} + \frac{1}{10}e^{t} - \cancel{1/5\,e^{-2t}} + 2/5\,e^{t} \\ \cancel{-4/5\,e^{-2t}} - \cancel{\frac{4}{10}e^{t}} - 1/5\,e^{-2t} + \cancel{2/5\,e^{t}} \end{pmatrix} = \begin{pmatrix} 1/2\,e^{t} \\ -e^{-2t} \end{pmatrix}$$

$$\vec{x} = c_1 \begin{pmatrix} -1 \\ 4 \end{pmatrix} e^{-3t} + c_2 \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{t} + \begin{pmatrix} 1/2\,e^{t} \\ -e^{-2t} \end{pmatrix}$$

7. Find the general solution of the system, and describe the general behavior of the system. (10 points each)

a. $t\vec{x}' = \begin{pmatrix}4 & -3\\ 8 & -6\end{pmatrix}\vec{x}$

$(4-\lambda)(-6-\lambda)+24 = \lambda^2+2\lambda-24+24 = \lambda^2+2\lambda = 0$ $\quad \lambda(\lambda+2)=0$ $\quad \lambda=0, \lambda=-2$

$\begin{pmatrix}4+2 & -3\\ 8 & -6+2\end{pmatrix} = \begin{pmatrix}6&-3\\8&-4\\2&-1\end{pmatrix}$ $\quad x_1 = \frac{1}{2}x_2$, $x_2=x_2$ $\quad \begin{pmatrix}1\\2\end{pmatrix}t^{-2}$ $\quad \vec{x} = c_1\begin{pmatrix}1\\2\end{pmatrix}t^{-2} + c_2\begin{pmatrix}3\\4\end{pmatrix}$

$\begin{pmatrix}4&-3\\8&-6\end{pmatrix} \Rightarrow$ $4x_1=3x_2$, $x_1=\frac{3}{4}x_2$, $x_2=x_2$ $\quad \begin{pmatrix}3\\4\end{pmatrix}t^0$

System → a multiple of $\begin{pmatrix}3\\4\end{pmatrix}$ vector.

b. $\vec{x}' = \begin{pmatrix}1&2\\-5&-1\end{pmatrix}\vec{x}$ $\quad (1-\lambda)(-1-\lambda)+10=0$ $\quad \lambda^2-1+10=\lambda^2+9=0$ $\quad \lambda=\pm 3i$

$\begin{pmatrix}1-3i & 2\\ -5 & -1-3i\end{pmatrix}$ $\quad -5x_1=(1+3i)x_2$, $x_1 = \frac{1+3i}{5}x_2$, $x_2=x_2$ $\quad \begin{pmatrix}1+3i\\5\end{pmatrix}$ $\quad \begin{pmatrix}1+3i\\5\end{pmatrix}(\cos 3t + i\sin 3t) =$

$\begin{pmatrix}\cos 3t + i\sin 3t + 3i\cos 3t - 3\sin 3t\\ 5\cos 3t + 5i\sin 3t\end{pmatrix} = c_1\begin{pmatrix}\cos 3t - 3\sin 3t\\5\cos 3t\end{pmatrix} + c_2\begin{pmatrix}\sin 3t + 3\cos 3t\\ 5\sin 3t\end{pmatrix} \vec{x}$

Spirals forever.

8. Find the fundamental matrix for $\vec{x}' = \begin{pmatrix}-1&-4\\1&-1\end{pmatrix}\vec{x}$. (8 points)

$\begin{pmatrix}-1-\lambda & -4\\ 1 & -1-\lambda\end{pmatrix}$ $\quad (-1-\lambda)(-1-\lambda)+4=0$, $\lambda^2+2\lambda+1+4=0$ $\quad \lambda^2+2\lambda+5=0$ $\quad \frac{-2\pm\sqrt{4-20}}{2} = \frac{-2\pm 4i}{2} = -1\pm 2i$

$\begin{pmatrix}-1+1-2i & -4\\1 & -1+1-2i\end{pmatrix} = \begin{pmatrix}-2i&-4\\1&-2i\end{pmatrix}$ $\quad x_1 = (-2i)x_2$, $x_2=x_2$ $\quad \begin{pmatrix}-2i\\1\end{pmatrix}(\cos 2t + i\sin 2t) =$

$\begin{pmatrix}-2i\cos 2t + 2\sin 2t\\ \cos 2t + i\sin 2t\end{pmatrix} \Rightarrow \boxed{e^{-t}\begin{pmatrix}+2\sin 2t & -2\cos 2t\\ \cos 2t & \sin 2t\end{pmatrix}}$

9. Consider the ODE $y' = \sqrt{t+y}$, $y(0) = 3$. Apply the specified method each for three steps with step size $h = 0.1$ to estimate the value of $y(0.3)$. Keep at least 4 decimal places.

a. Use Euler's method (12 points)

$y_0 = 3$ $\quad t_0=0$ $\quad m=\sqrt{0+3}=\sqrt{3}$ $\quad y_1=\sqrt{3}(.1)+3 = 3.173205081$

$y_1 = 3.1732$ $\quad t_1=.1$ $\quad m=\sqrt{.1+3.1732}=1.8092$ $\quad y_2 = 1.8092*.1+3.1732 = 3.3541\ldots$

$y_2 = 3.3541\ldots$ $\quad t_2=.2$ $\quad m=\sqrt{.2+3.3541}=1.8852\ldots$ $\quad y_2 = 1.8852*.1+3.3541\ldots = 3.54262\ldots$

$\boxed{y_3 = 3.7300 \quad t_3=.3}$

$(.3, 3.54262)$

b. The modified Euler's method $y_{n+1} = y_n + hf\left[t_n + \frac{1}{2}h, y_n + \frac{1}{2}h\overbrace{f(t_n, y_n)}^{m}\right]$ (12 points)

$y_0 = 3 \quad t_0 = 0 \qquad m = \sqrt{0+3} = \sqrt{3} \qquad y_1 = 3 + (.1)(\sqrt{.05 + 3 + .05\sqrt{3}}) = 3.177104561$

$y_1 = 3. \qquad t = .1 \quad m_2 = \sqrt{.1 + 3.177...} = 1.8107... \quad y_2 = 3.1771... + (.1)\sqrt{.15 + 3.1771... + .05(1.8107)}$

$= 3.36197...$

$y_2 = 3. \qquad t_2 = .2 \quad m_2 = \sqrt{.2 + 3.36197...} = 1.8873... \quad y_3 = 3.361972... + .1\sqrt{.25 + 3.36197... + .05(1.8873)}$

$= 3.55449...$

$y_3 = 3. \qquad t_3 = .3$

$(.3, 3.554)$

c. Runge-Kutta method $y_{n+1} = y_n + h\left(\frac{k_{n1} + 2k_{n2} + 2k_{n3} + k_{n4}}{6}\right)$, (12 points) (one step) $h = .1$

$k_{n1} = f(t_n, y_n), k_{n2} = f\left(t_n + \frac{1}{2}h, y_n + \frac{1}{2}hk_{n1}\right), \quad k_{n3} = f\left(t_n + \frac{1}{2}h, y_n + \frac{1}{2}hk_{n2}\right),$

$k_{n4} = f(t_n + h, y_n + hk_{n3})$

$y_0 = 3 \quad t_0 = 0 \qquad k_{n1} = \sqrt{0+3} = \sqrt{3} \qquad k_{n2} = \sqrt{.05 + 3 + .05(\sqrt{3})} = 1.771045607$

$k_{n3} = \sqrt{.05 + 1.771045607 * .05 + 3} = 1.77159597$

$k_{n4} = \sqrt{.1 + 3 + .1(1.77159597)} = 1.81029...$

$y_1 = 3 + \frac{.1}{6}\left(\sqrt{3} + 2 * 1.771045607 + 2 * 1.77159597 + 1.81029\right)$

$= 3.177127111$

$(.1, 3.1771)$

d. Compare the results of the three methods. (5 points)

answers will vary

be sure to be comparing comparable steps

10. Use power series to solve $y'' + xy' + 2y = 0$ centered at $x_0 = 0$. Write out at least 4 terms of each solution. (12 points)

$$\sum_{n=2}^{\infty} n(n-1)a_n x^{n-2} + x\sum_{n=1}^{\infty} na_n x^{n-1} + 2\sum_{n=0}^{\infty} a_n x^n = 0$$

$$\sum_{n=2}^{\infty} n(n-1)a_n x^{n-2} + \sum_{n=1}^{\infty} na_n x^n + \sum_{n=0}^{\infty} 2a_n x^n = 0$$

$$\sum_{n=0}^{\infty} (n+2)(n+1)a_{n+2} x^n + \sum_{n=1}^{\infty} n a_n x^n + \sum_{n=0} 2a_n x^n = 0$$

$$(2)(1)a_2(1) + \sum_{n=1}^{\infty}(n+2)(n+1)a_{n+2}x^n + \sum_{n=1}^{\infty} n a_n x^n + \sum_{n=1}^{\infty} 2a_n x^n + 2a_0(1) = 0$$

$$2a_2 + 2a_0 = 0 \qquad \sum_{n=1}^{\infty}\Big[(n+2)(n+1)a_{n+2} + \underbrace{na_n + 2a_n}_{(n+2)a_n}\Big]x^n = 0$$

$$a_2 = -a_0$$

$$a_{n+2} = \frac{-\cancel{(n+2)}a_n}{\cancel{(n+2)}(n+1)} = \frac{-a_n}{n+1}$$

$$a_3 = \frac{-a_1}{2}$$

$$a_4 = \frac{-a_2}{3} = \frac{-(-a_0)}{3} = \frac{1}{3}a_0$$

$$a_5 = \frac{-a_3}{4} = -\frac{1}{4}\left(-\frac{1}{2}a_1\right) = \frac{1}{8}a_1$$

$$a_6 = \frac{-a_4}{5} = -\frac{1}{5}\left(\frac{1}{3}a_0\right) = \frac{-1}{15}a_0$$

$$a_7 = \frac{-a_5}{6} = -\frac{1}{6}\left(\frac{1}{8}a_1\right) = -\frac{1}{48}a_1$$

$$y(x) = a_0\left(1 - x^2 + \frac{1}{3}x^4 - \frac{1}{15}x^6 + \dots\right) + a_1\left(x - \frac{1}{2}x^2 + \frac{1}{8}x^5 - \frac{1}{48}x^7 + \dots\right)$$

11. For $x^2(1-x)^2y'' + 2xy' + 4y = 0$, determine the location of any singular points, and for each one, classify it as regular or irregular. (6 points)

$$y'' + \frac{2\cancel{x}}{x^{\cancel{2}}(1-x)^2}y' + \frac{4}{x^2(1-x)^2}y = 0$$

$$\frac{2}{x(1-x)^2}$$

$x=0$ regular singular point

$x=1$ irregular singular point

12. Use the attached table to find the inverse Laplace transform of each function. (6 points each)

a. $F(s) = \frac{8s^2-4s+12}{s(s^2+1)}$

$$\frac{A}{s} + \frac{Bs+C}{s^2+1} = \frac{As^2+A+Bs^2+Cs}{s(s^2+1)} \Rightarrow$$

$A+B=8 \quad B=-4$

$C=-4$

$A=12$

$$\frac{12}{s} - \frac{4(s+1)}{s^2+1}$$

$$\boxed{f(t) = 12 - 4\cos t - 4\sin t}$$

b. $F(s) = \frac{1}{s^3(s^2+4)} = \frac{1}{s^3} \cdot \frac{1}{s^2+4}$

$$\boxed{f(t) = \int_0^t \frac{1}{2}(t-\tau)^2 \sin(2\tau)\,d\tau}$$ answers may vary

c. $F(s) = \frac{(s-3)e^{-1}}{s^2-4s+3}$ $\quad \frac{\cancel{(s-3)}e^{-1}}{\cancel{(s-3)}(s-1)} = \frac{e^{-1}}{s-1}$

$$\boxed{f(t) = e^{-1}(e^{t})} = e^{t-1}$$

13. Use a Laplace transform to (fully) solve $y'' - 2y' + 4y = 0, y(0) = 2, y'(0) = 0$. (12 points)

$$s^2F(s) - s(2) - 0 - 2(sF(s) - 2) + 4(F(s)) = 0$$

$$F(s) = (s^2 - 2s + 4) - 2s + 4 = 0$$

$$F(s) = \frac{2s-4}{s^2-2s+4} = \frac{2s-4}{(s^2-2s+1)+3} = \frac{2s-4}{(s-1)^2+3} = \frac{2s}{(s-1)^2+3} - \frac{4}{(s-1)^2+3}$$

$$= \frac{2(s-1)}{(s-1)^2+3} - \frac{2}{(s-1)^2+3}$$

$$y(t) = 2e^t\cos(\sqrt{3}t) - \frac{2}{\sqrt{3}}e^t\sin(\sqrt{3}t)$$

14. Classify the following differential equations by i) linearity, ii) order, iii) ordinary or partial. (15 points)

a. $u_t + uu_x = 1 + u_{xx}$

nonlinear, partial, second order

b. $y''' - 3y'' + 2y' = 0$

linear, ordinary, 3rd order

c. $\frac{d^3y}{dt^3} + t\frac{dy}{dt} + (\cos^2 y)t = t^3$

nonlinear, ordinary, 3rd order

15. Solve $2y' - y = e^{\frac{t}{3}}$ by the method of integrating factors. (10 points)

$y' - \frac{1}{2}y = \frac{1}{2}e^{t/3}$ $\qquad \mu = e^{\int -1/2\, dt} = e^{-1/2 t}$

$e^{-1/2t}y' - \frac{1}{2}e^{-1/2t}y = \frac{1}{2}e^{-1/6t}$

$\int (e^{-1/2t}y)' = \int \frac{1}{2}e^{-1/6t}\,dt \Rightarrow \left(e^{-1/2t}y = -3e^{-1/6t} + C\right)e^{1/2t}$

$\boxed{y(t) = -3e^{t/3} + Ce^{1/2t}}$

16. A tank has pure water flowing into it at 5 L/min. The contents of the tank are kept thoroughly mixed, and the contents flow out at 5 L/min. Salt is added to the tank at the rate of 0.1 kg/min. Initially, the tank contains 10 kg of salt in 300 L of water. How much salt is in the tank after 30 minutes? (12 points)

$\frac{dQ}{dt} = \text{Rate}_{in} - \text{Rate}_{out}$

$\text{Rate}_{in} = \frac{5L}{min} \cdot \frac{0kg}{L} = 0 + .1 kg/min$

$\text{Rate}_{out} = \frac{5\cancel{L}}{min} \cdot \frac{Q kg}{300\cancel{L}} = \frac{Q}{60}$

$\frac{dQ}{dt} = .1 - \frac{Q}{60} \Rightarrow -\frac{1}{60}(Q-6)$

$Q(0) = 10$

$Q_0 = 4$

$\int \frac{dQ}{Q-6} = \int -\frac{1}{60}\,dt \qquad \ln|Q-6| = -\frac{1}{60}t + C$

$Q - 6 = e^{-1/60t} \cdot Q_0$

$Q(t) = Q_0 e^{-1/60t} + 6 = 4e^{-1/60t} + 6$

$Q(30) = 4e^{-1/60 \cdot 30} + 6 = 8.426$ kg.

17. Find the general solution for each of the following: (8 points each)

a. $y'' + 2y' + 2y = 0$

$r^2 + 2r + 2 = 0$

$\frac{-2 \pm \sqrt{4-8}}{2} = \frac{-2 \pm 2i}{2} = -1 \pm i$

$$\boxed{y(t) = C_1 e^{-t}\cos t + C_2 e^{-t}\sin t}$$

b. $y^{IV} - 8y' = 0$

$$\boxed{y(t) = C_1 + C_2 e^{2t} + C_3 e^{-t}\cos(\sqrt{3}t) + C_4 e^{-t}\sin(\sqrt{3}t)}$$

$r^4 - 8r = 0$

$r(r^3 - 8) = 0$

$r(r-2)(r^2 + 2r + 4) = 0$

$r = 0,\ r = 2$

$\frac{-2 \pm \sqrt{4-16}}{2} = \frac{-2 \pm 2\sqrt{3}i}{2} = -1 \pm \sqrt{3}i$

18. What Ansatz would you need to solve for the given forcing function $F(t)$ and the specified solutions $y_1(t)$, $y_2(t)$ to the second order ODE. (4 points each)

	$y_1(t)$	$y_2(t)$	$F(t)$	Ansatz
a.	$\sin t$	$\cos t$	$9 \sin 2t$	$A\sin 2t + B\cos 2t$
b.	e^{-2t}	te^{-2t}	$\frac{1}{4}e^{-2t}$	$At^2 e^{-2t}$
c.	e^t	e^{-2t}	$3 \cos t$	$B\cos t + A\sin t$
d.	e^t	$\frac{1}{t}$	$t^3 - 6$	$At^3 + Bt^2 + Ct + D$

19. What are the conditions for a resonance phenomenon to arise in a solution to a second order ODE? Give an example of an equation with such a solution. (6 points)

generally undamped system (for true mathematical resonance) with a forcing function at the same frequency as the natural frequency.

$y'' + y = \sin t$

answers will vary

$Laplace\ transforms - Table$			
$f(t) = L^{-1}\{F(s)\}$	$F(s)$	$f(t) = L^{-1}\{F(s)\}$	$F(s)$
$a \quad t \geq 0$	$\frac{a}{s} \qquad s > 0$	$\sin \omega t$	$\frac{\omega}{s^2 + \omega^2}$
$at \quad t \geq 0$	$\frac{a}{s^2}$	$\cos \omega t$	$\frac{s}{s^2 + \omega^2}$
e^{-at}	$\frac{1}{s+a}$	$\sin(\omega t - \theta)$	$\frac{s \sin\theta + \omega \cos\theta}{s^2 + \omega^2}$
te^{-at}	$\frac{1}{(s+a)^2}$	$\cos(\omega t + \theta)$	$\frac{s \cos\theta - \omega \sin\theta}{s^2 + \omega^2}$
$\frac{1}{2}t^2e^{-at}$	$\frac{1}{(s+a)^3}$	$t \sin \omega t$	$\frac{2\omega s}{(s^2 + \omega^2)^2}$
$\frac{1}{(n-1)!}t^{n-1}e^{-at}$	$\frac{1}{(s+a)^n}$	$t \cos \omega t$	$\frac{s^2 - \omega^2}{(s^2 + \omega^2)^2}$
e^{at}	$\frac{1}{s-a} \qquad s > a$	$\sinh \omega t$	$\frac{\omega}{s^2 - \omega^2} \qquad s > \vert\omega\vert$
te^{at}	$\frac{1}{(s-a)^2}$	$\cosh \omega t$	$\frac{s}{s^2 - \omega^2} \qquad s > \vert\omega\vert$
$\frac{1}{b-a}(e^{-at} - e^{-bt})$	$\frac{1}{(s+a)(s+b)}$	$e^{-at} \sin \omega t$	$\frac{\omega}{(s+a)^2 + \omega^2}$
$\frac{1}{a^2}[1 - e^{-at}(1 + at)]$	$\frac{1}{s(s+a)^2}$	$e^{-at} \cos \omega t$	$\frac{s+a}{(s+a)^2 + \omega^2}$
t^n	$\frac{n!}{s^{n+1}} \qquad n = 1,2,3..$	$e^{at} \sin \omega t$	$\frac{\omega}{(s-a)^2 + \omega^2}$
$t^n e^{at}$	$\frac{n!}{(s-a)^{n+1}} \qquad s > a$	$e^{at} \cos \omega t$	$\frac{s-a}{(s-a)^2 + \omega^2}$
$t^n e^{-at}$	$\frac{n!}{(s+a)^{n+1}} \qquad s > a$	$1 - e^{-at}$	$\frac{a}{s(s+a)}$
$\sqrt{t}$	$\frac{\sqrt{\pi}}{2s^{3/2}}$	$\frac{1}{a^2}(at - 1 + e^{-at})$	$\frac{1}{s^2(s+a)}$
$\frac{1}{\sqrt{t}}$	$\sqrt{\frac{\pi}{s}} \qquad s > 0$	$f(t - t_1)$	$e^{-t_1 s}F(s)$
$g(t) \cdot p(t)$	$G(s) \cdot P(s)$	$f_1(t) \pm f_2(t)$	$F_1(s) \pm F_2(s)$
$\int f(t)dt$	$\frac{F(s)}{s} + \frac{f^{-1}(0)}{s}$	$\delta(t)\ unit\ impulse$	$1 \qquad all\ s$
$\frac{df}{dt}$	$sF(s) - f(0)$	$\frac{d^2f}{df^2}$	$s^2F(s) - sf(0) - f'(0)$
$\frac{d^nf}{dt^n}$	$s^nF(s) - s^{n-1}f(0) - s^{n-2}f'(0) - s^{n-3}f''(0) - \cdots - f^{n-1}(0)$		

Table of Laplace Transforms

$f(t)=\mathfrak{L}^{-1}\{F(s)\}$	$F(s)=\mathfrak{L}\{f(t)\}$	$f(t)=\mathfrak{L}^{-1}\{F(s)\}$	$F(s)=\mathfrak{L}\{f(t)\}$
1. 1	$\frac{1}{s}$	2. e^{at}	$\frac{1}{s-a}$
3. $t^n,\quad n=1,2,3,\ldots$	$\frac{n!}{s^{n+1}}$	4. $t^p,\ p>-1$	$\frac{\Gamma(p+1)}{s^{p+1}}$
5. $\sqrt{t}$	$\frac{\sqrt{\pi}}{2s^{\frac{3}{2}}}$	6. $t^{n-\frac{1}{2}},\quad n=1,2,3,\ldots$	$\frac{1\cdot3\cdot5\cdots(2n-1)\sqrt{\pi}}{2^n s^{n+\frac{1}{2}}}$
7. $\sin(at)$	$\frac{a}{s^2+a^2}$	8. $\cos(at)$	$\frac{s}{s^2+a^2}$
9. $t\sin(at)$	$\frac{2as}{(s^2+a^2)^2}$	10. $t\cos(at)$	$\frac{s^2-a^2}{(s^2+a^2)^2}$
11. $\sin(at)-at\cos(at)$	$\frac{2a^3}{(s^2+a^2)^2}$	12. $\sin(at)+at\cos(at)$	$\frac{2as^2}{(s^2+a^2)^2}$
13. $\cos(at)-at\sin(at)$	$\frac{s(s^2-a^2)}{(s^2+a^2)^2}$	14. $\cos(at)+at\sin(at)$	$\frac{s(s^2+3a^2)}{(s^2+a^2)^2}$
15. $\sin(at+b)$	$\frac{s\sin(b)+a\cos(b)}{s^2+a^2}$	16. $\cos(at+b)$	$\frac{s\cos(b)-a\sin(b)}{s^2+a^2}$
17. $\sinh(at)$	$\frac{a}{s^2-a^2}$	18. $\cosh(at)$	$\frac{s}{s^2-a^2}$
19. $e^{at}\sin(bt)$	$\frac{b}{(s-a)^2+b^2}$	20. $e^{at}\cos(bt)$	$\frac{s-a}{(s-a)^2+b^2}$
21. $e^{at}\sinh(bt)$	$\frac{b}{(s-a)^2-b^2}$	22. $e^{at}\cosh(bt)$	$\frac{s-a}{(s-a)^2-b^2}$
23. $t^n e^{at},\quad n=1,2,3,\ldots$	$\frac{n!}{(s-a)^{n+1}}$	24. $f(ct)$	$\frac{1}{c}F\left(\frac{s}{c}\right)$
25. $u_c(t)=u(t-c)$ Heaviside Function	$\frac{e^{-cs}}{s}$	26. $\delta(t-c)$ Dirac Delta Function	e^{-cs}
27. $u_c(t)f(t-c)$	$e^{-cs}F(s)$	28. $u_c(t)g(t)$	$e^{-cs}\mathfrak{L}\{g(t+c)\}$
29. $e^{ct}f(t)$	$F(s-c)$	30. $t^n f(t),\quad n=1,2,3,\ldots$	$(-1)^n F^{(n)}(s)$
31. $\frac{1}{t}f(t)$	$\int_s^\infty F(u)\,du$	32. $\int_0^t f(v)\,dv$	$\frac{F(s)}{s}$
33. $\int_0^t f(t-\tau)g(\tau)\,d\tau$	$F(s)G(s)$	34. $f(t+T)=f(t)$	$\frac{\int_0^T e^{-st}f(t)\,dt}{1-e^{-sT}}$
35. $f'(t)$	$sF(s)-f(0)$	36. $f''(t)$	$s^2F(s)-sf(0)-f'(0)$
37. $f^{(n)}(t)$	$s^nF(s)-s^{n-1}f(0)-s^{n-2}f'(0)\cdots-sf^{(n-2)}(0)-f^{(n-1)}(0)$		