MAT 212, Exam #1, Spring 2017 Name KEY

Instructions: Show all work. Answers without work required to obtain the solution will not receive full credit. Some questions may contain multiple parts: be sure to answer all of them. Give exact answers unless specifically asked to estimate.

1. Verify that $y(x) = \ln(x + C)$ is a solution to the differential equation $e^y y' = 1, y(0) = 0$. (6 points)

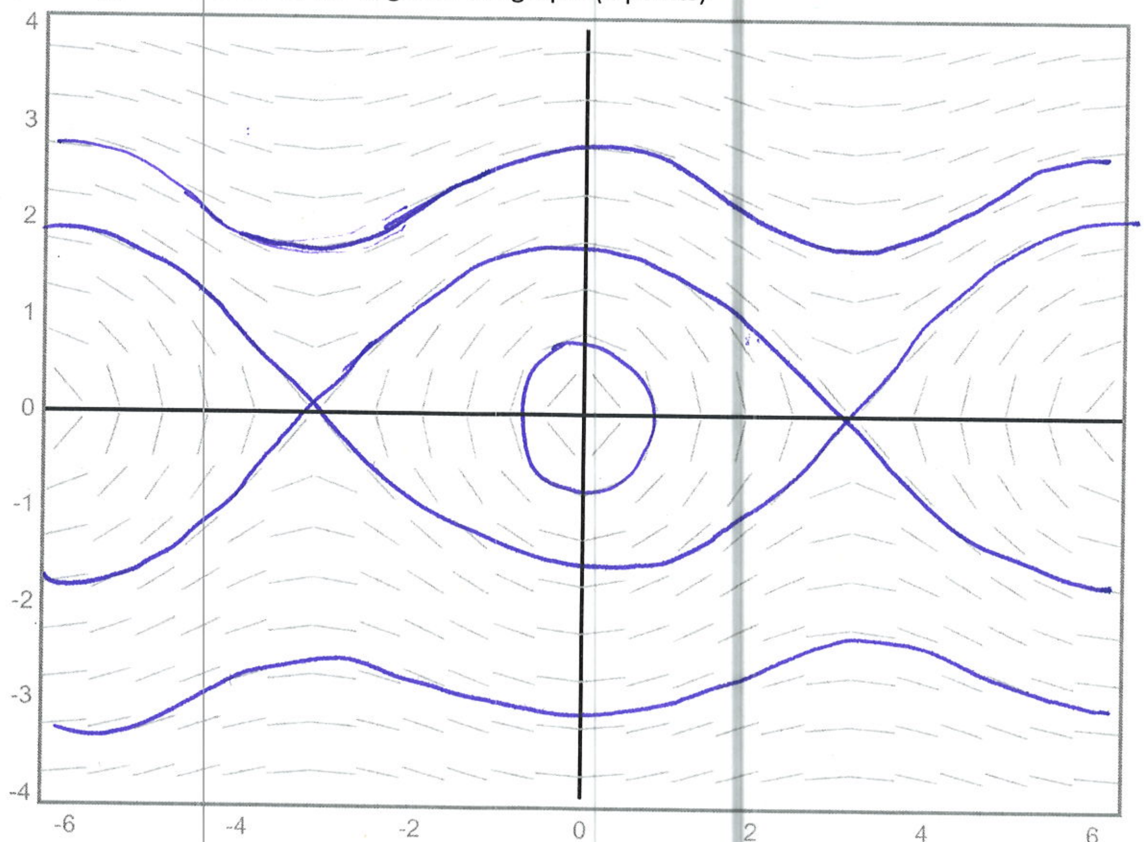
$y' = \frac{1}{x+C}$

$e^y = e^{\ln(x+c)} = x+c$

$e^y y' = (\cancel{x+c})\frac{1}{\cancel{x+c}} = 1$ ✓

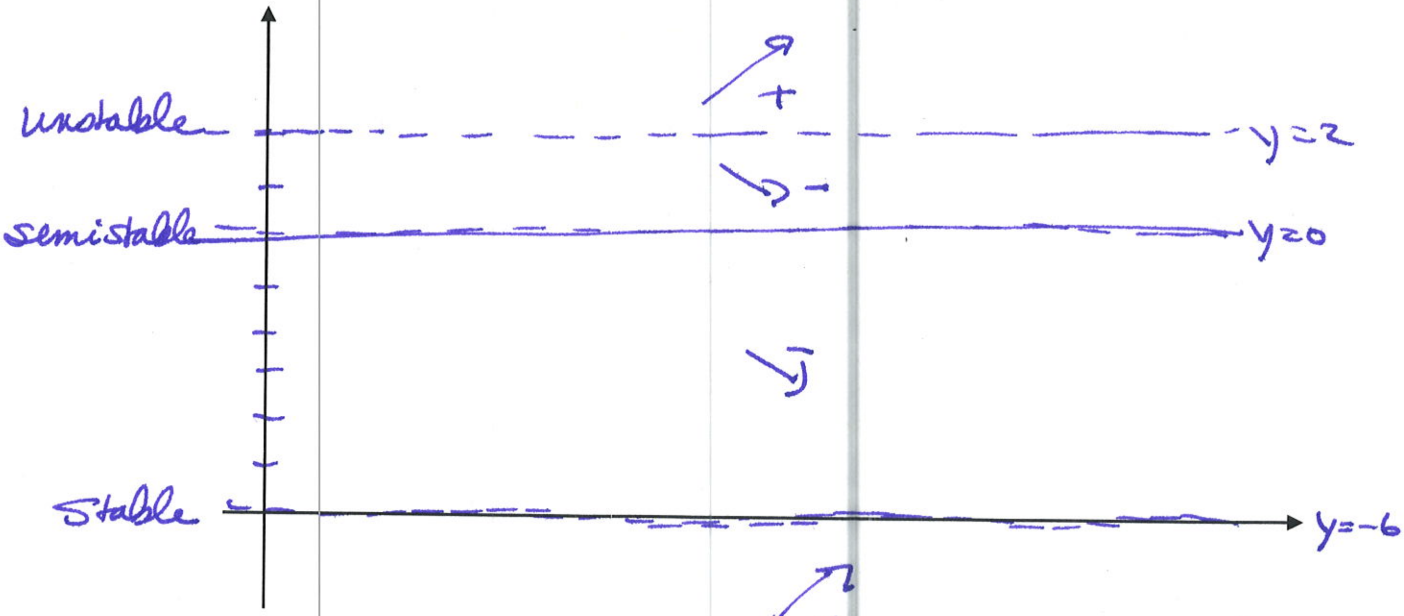
yes, it is a solution

2. Shown below is the slope field for an undamped pendulum. Plot at least three sample trajectories with different behaviors. Track the path forward and backward in time (so that the path begins and ends on the edges of the graph. (6 points)

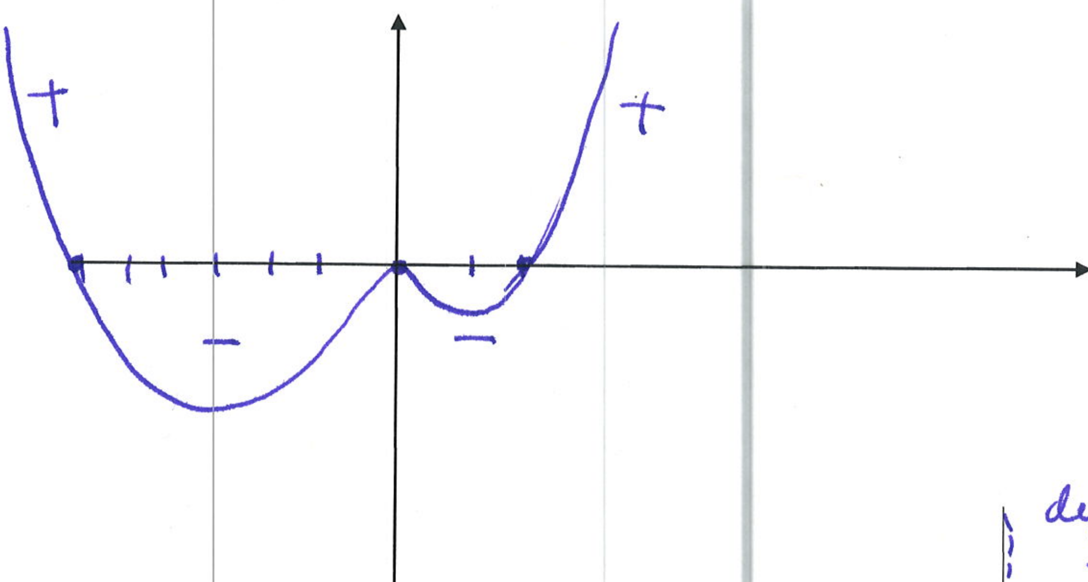


3. Sketch the phase plane, and use that to draw the slope field for the autonomous equation $y' = y^2(y-2)(y+6)$. Label each equilibrium and classify it as stable, semi-stable or unstable. (12 points)

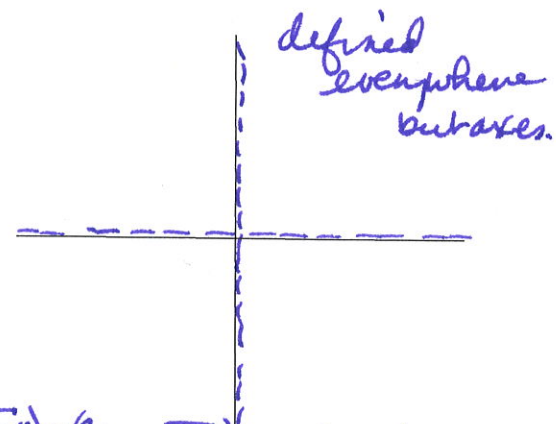
Direction Field



Phase Plane



4. Sketch the region in the plane where a solution to the ODE $y^2(xy' + y)\sqrt{1+x^4} = x$ is guaranteed to exist. Be sure to check all conditions and show your work. (12 points)



$$xy' + y = \frac{x}{\sqrt{1+x^4}\,y^2}$$

$$xy' = \frac{x}{y^2\sqrt{1+x^4}} - y$$

$$y' = \frac{1}{y^2\sqrt{1+x^4}} - \frac{y}{x} = \frac{x - y^3\sqrt{1+x^4}}{xy^2\sqrt{1+x^4}} = f(x,y)$$

$xy^2\sqrt{1+x^4} \neq 0$

$x \neq 0 \quad y \neq 0 \quad \sqrt{1+x^4} \neq 0$

$$f_y(x,y) = \frac{(3y^2\sqrt{1+x^4})(xy^2\sqrt{1+x^4}) - (2yx\sqrt{1+x^4})(x - y^3\sqrt{1+x^4})}{x^2y^4(1+x^4)}$$

$x \neq 0, \ y \neq 0, \ 1+x^4 \neq 0$

5. Solve $x^2y' = 1 - x^2 + y^2 - x^2y^2$ by separation of variables. [Hint: Factor by grouping.] (12 points)

$1(1-x^2) + y^2(1-x^2)$

$x^2y' = (1+y^2)(1-x^2)$

$\frac{dy}{1+y^2} = \frac{1-x^2}{x^2}\,dx$

$\int \frac{dy}{1+y^2} = \int x^{-2} - 1\,dx$

$\arctan(y) = \frac{-1}{x} - x + C$

6. Classify each differential equation as i) linear or nonlinear, ii) ordinary or partial, iii) its order. (3 points each)

a. $x^2y'' + 5xy' + 4y = 0$

linear, ordinary, second order

b. $\frac{\partial u}{\partial x} + \frac{\partial^2 u}{\partial x^2} = yu$

linear, partial, second order

c. $\frac{d^3y}{dt^3} + 2\cos t\,\frac{d^2y}{dt^2} + \left(\frac{dy}{dt}\right)^2 = e^t$

nonlinear, ordinary, third order

d. $u_{xx} + u_y = \ln(u)$

nonlinear, partial, second order

7. A 400 gal tank initially contains 100 gal of brine containing 50 lbs of salt. Brine containing 1 lbs of salt per gal enters the tank at a rate of 5 gal/s, and the well-mixed brine flows out of the tank at the rate of 3 gal/s. How much salt will the tank contain when the tank is completely full? (10 points)

$A(0)=50$

$$\frac{dA}{dt} = 1\cdot 5 - \frac{3A}{100+2t}$$

$$\mu = e^{\int \frac{3}{100+2t}dt} = e^{\frac{3}{2}\ln(100+2t)} = e^{\ln(100+2t)^{3/2}} = (100+2t)^{3/2}$$

$$A' + \frac{3}{100+2t}A = 5$$

$$(100+2t)^{3/2}A' + 3(100+2t)^{1/2}A = 5(100+2t)^{3/2}$$

$$\int\left((100+2t)^{3/2}A\right)' = \int 5(100+2t)^{3/2}$$

$$(100+2t)^{3/2}A = \cancel{5}\frac{2}{\cancel{5}}(100+2t)^{5/2}\frac{1}{\cancel{2}} + C$$

$$A = (100+2t) + \frac{C}{(100+2t)^{3/2}}$$

$$50 = 100 + \frac{C}{1000}$$

$$-50 = \frac{C}{1000} \rightarrow C = -50{,}000$$

$$A = (100+2t) + \frac{-50{,}000}{(100+2t)^{3/2}}$$

full is 400 gal. $100+2t=400$

$t=150$

$\boxed{793.75 \text{ lbs}}$

8. Use the method of integrating factors to find the particular solution for $xy' - 3y = 2x^2e^x$. (10 points)

$$y' - \frac{3}{x}y = 2xe^x \qquad \mu = e^{\int -\frac{3}{x}dx} = e^{-3\ln x} = e^{\ln x^{-3}} = x^{-3}$$

$$x^{-3}y' - 3x^{-4}y = \frac{2}{x^2}e^x$$

$$\int (x^{-3}y)' = \int \frac{2}{x^2}e^x dx$$

$$x^{-3}\cdot y = \int \frac{2}{x^2}e^x dx$$

$$y = x^3\int \frac{2}{x^2}e^x dx$$

or $$y = x^3\int_0^x \frac{2}{t^2}e^t dt$$

9. Rewrite the Bernoulli equation $x^2y' + 2xy = 5y^4$ as a linear equation. [You do not need to solve.] (8 points)

$n=4$

$$(1-n)y^{-n}$$

$$(1-4)y^{-4} - (-3)y^{-4}$$

$$z = y^{-3}$$

$$z' = -3y^{-4}y'$$

$$-3x^2y^{-4}y' - 6xy^{-3} = -15$$

$$x^2z' - 6xz = -15$$

$$z' - \frac{6}{x}z = \frac{-15}{x^2}$$

10. Verify that the equation $(2x + y^2)dx + (2xy)dy = 0$ is exact. Then find the general solution. (10 points)

$M = 2x + y^2 \qquad N = 2xy$

$\frac{\partial M}{\partial y} = 2y \qquad \frac{\partial N}{\partial x} = 2y$

$\varphi = x^2 + xy^2 + K$

$\int M\,dx = x^2 + xy^2 + f(y)$

$\int N\,dy = x^2 y + g(x)$

11. Solve the homogeneous equation $(x + 2y)y' = y$, and state its order. (10 points)

$y' = \frac{y}{x+2y}$

$y = vx$

$y' = v'x + v$

$v = \frac{y}{x}$

$v'x + v = \frac{vx}{x + 2vx}$

$v'x + v = \frac{x(v)}{x(1+2v)}$

$v'x + v = \frac{v}{1+2v}$

$-v \qquad -v$

$v'x = \frac{v}{1+2v} - \frac{v(1+2v)}{1+2v}$

$v'x = \frac{\cancel{v} - \cancel{v} - 2v^2}{1+2v} = \frac{-2v^2}{1+2v}$

$\left(\frac{1+2v}{2v^2}\right) dv = \int -\frac{1}{x}dx = \int\left(\frac{1}{2}v^{-2} + \frac{1}{v}\right) dv$

$-\ln x + C = \frac{-1}{2v} + \ln v$

$\cancel{-\ln x} + C = \frac{-x}{2y} + \ln\left(\frac{y}{x}\right) = -\frac{x}{2y} + \ln y \cancel{-\ln x}$

$\boxed{-\frac{x}{2y} + \ln y = C}$

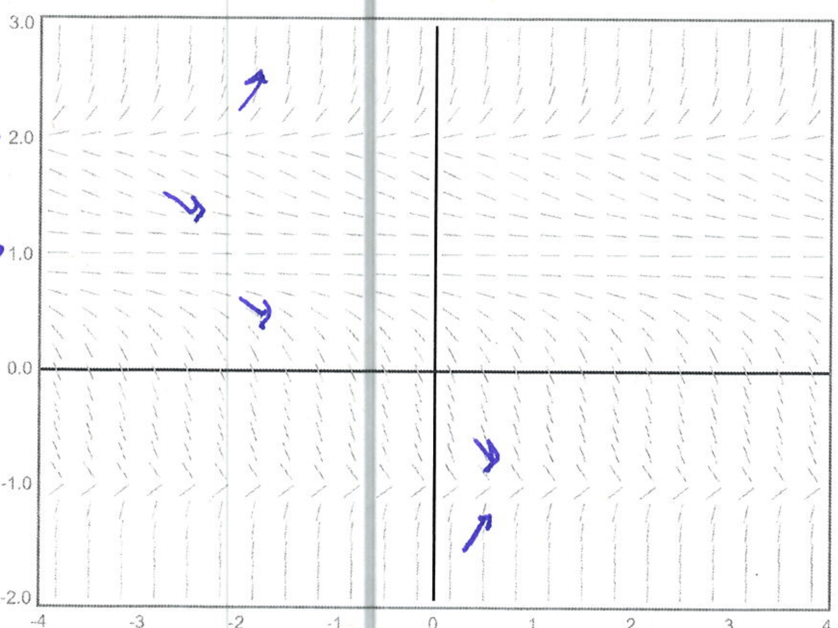
12. Consider the slope field shown below. If the equilibria are assumed to be integer values, write a differential equation that can produce the field. Characterize each equilibria as stable, unstable or semistable. (6 points)

$y' = (y-2)(y-1)^2(y+1)$

unstable → 2.0

Semistable → 1.0

Stable → -1.0



13. Describe the conditions in a population model needed to label an equilibrium as a carrying capacity. (4 points)

the value of equilibrium > 0
and increases below, decreases above;
i.e. equilibrium is stable

14. Use Euler's Method for 5 steps to approximate the solution of $xy' = y^2, y(1) = 1$ at the point $y(2)$. (14 points)

$y' = \frac{y^2}{x}$ $\quad \Delta x = \frac{2-1}{5} = \frac{1}{5} = .2$

$x_0 = 1 \quad y_0 = 1 \quad m_0 = \frac{1^2}{1} = 1 \quad y_1 = 1 + \frac{1}{5}(1) = 1.2$

$x_1 = 1.2 \quad y_1 = 1.2 \quad m_1 = \frac{1.2^2}{1.2} = 1.2 \quad y_2 = 1.2 + \frac{1}{5}(1.2) = 1.44$

$x_2 = 1.4 \quad y_2 = 1.44 \quad m_2 = \frac{1.44^2}{1.4} = 1.48 \quad y_3 = 1.44 + \frac{1}{5}(1.48) = 1.736$

$x_3 = 1.6 \quad y_3 = 1.736 \quad m_3 = \frac{1.736^2}{1.6} = 1.884 \quad y_4 = 1.736 + \frac{1}{5}(1.884) = 2.113\ldots$

$x_4 = 1.8 \quad y_4 = 2.113 \quad m_4 = \frac{2.113^2}{1.8} = 2.48 \quad y_5 = 2.113 + \frac{1}{5}(2.48) = 2.609$

$\boxed{x_5 = 2 \quad y_5 = 2.609}$

$y(2) \approx 2.609$