MAT 212, Exam #3, Spring 2017 Name KEY

Instructions: Show all work. Answers without work required to obtain the solution will not receive full credit. Some questions may contain multiple parts: be sure to answer all of them. Give exact answers unless specifically asked to estimate.

1. Use the definition of the Laplace transform $\mathcal{L}\{f(t)\} = \int_0^\infty e^{-st} f(t)dt$ to find $\mathcal{L}\{1 + \cosh 5t\}$ (8 points)

$$\int_0^\infty e^{-st}(1+\underbrace{\cosh 5t}_{\frac{e^{-5t}+e^{5t}}{2}})\,dt = \int_0^\infty e^{-st} + \tfrac{1}{2}e^{-t(s+5)} + \tfrac{1}{2}e^{-t(s-5)}\,dt$$

$$-\frac{1}{s}e^{-st} + \frac{1}{2}\cdot\frac{-1}{s+5}e^{-t(s+5)} + \frac{1}{2}\cdot\frac{-1}{s-5}e^{-t(s-5)}\Big|_0^\infty =$$

$$0+\frac{1}{s} - 0 + \frac{1}{2(s+5)} - 0 + \frac{1}{2(s-5)} = \frac{1}{s} + \frac{1}{2}\left[\frac{s-5+s+5}{s^2-25}\right] =$$

$$\frac{1}{s} + \frac{s}{s^2-25}$$

2. Use the table of Laplace transforms to find Laplace transforms or inverse Laplace transforms as indicated. (3 points each)

a. $\mathcal{L}\{(1+t)^2\} = \mathcal{L}\{1+2t+t^2\} =$

$$\frac{1}{s} + \frac{2}{s^2} + \frac{2}{s^3}$$

b. $\mathcal{L}\{te^t\}$

$$-\frac{d}{ds}\left[\frac{1}{s-1}\right] = \frac{+1}{(s-1)^2}$$

c. $\mathcal{L}\{e^{-2t}\sin 3\pi t\}$

$$\frac{3\pi}{(s+2)^2+9\pi^2}$$

d. $\mathscr{L}\{\sin t * t^2 e^{2t}\} = \mathcal{L}\left\{\int_0^t \sin(t-\tau)\cdot \tau^2 e^{2\tau}\, d\tau\right\}$

$$\frac{1}{s^2+1}\cdot \frac{d}{ds}\left\{\frac{d}{ds}\left[\frac{1}{s-2}\right]\right\} =$$

$$\frac{1}{s^2+1}\cdot \frac{2}{(s-2)^3}$$

e. $\mathscr{L}\left\{\frac{\sin t}{t}\right\}$

$$\arctan\left(\frac{1}{s}\right)$$

f. $f(t) = \begin{cases} \cos \pi t, & 0 \le t < 2 \\ t & t \ge 2 \end{cases}$, $\mathscr{L}\{f(t)\}$

$$\cos \pi t - [(t - \cos \pi t)]u(t-2)$$

$$t+2 - \cos\pi(t+2)$$

$$\frac{s}{s^2+\pi^2} - \left[\frac{1}{s^2} + \frac{2}{s} - \frac{s\cos 2 - \pi \sin 2}{s^2+\pi^2}\right] e^{-2s}$$

g. $\mathscr{L}\left\{\frac{1}{2}\int_0^t (t-\tau)^3 \sin 2\tau \, d\tau\right\}$

$$\frac{1}{\cancel{2}}\cdot \frac{6}{s^4}\cdot \frac{\cancel{2}}{s^2+4} = \frac{6}{s^4(s^2+4)}$$

h. $\mathscr{L}\{\delta(t-1)\}$

$$e^{-s}$$

i. $\mathscr{L}^{-1}\left\{\frac{1}{2} - \frac{2}{s^5}\right\}$

$$\frac{1}{2}\delta(t) - \frac{1}{12}t^4$$

j. $\mathcal{L}^{-1}\left\{\frac{9-17s}{s^2+81}\right\}$ $= \frac{9}{s^2+81} - \frac{17s}{s^2+81}$

$\sin 9t - 17\cos 9t$

k. $\mathcal{L}^{-1}\left\{\frac{1}{s(s^2+4)}\right\}$ $\frac{1}{s} \cdot \frac{1}{s^2+4}$

$\frac{1}{2}\int_0^t \sin 2\tau \, d\tau$

l. $\mathcal{L}^{-1}\left\{\frac{1}{s^2(s^2-1)}\right\}$ $\frac{1}{s^2} \cdot \frac{1}{s^2-1}$

$\int_0^t (t-\tau)\sinh \tau \, d\tau$

m. $\mathcal{L}^{-1}\left\{\frac{e^{-\pi s}}{s^2+1}\right\}$ $e^{-\pi s} \cdot \frac{1}{s^2+1}$

$\sin(t-\pi)\, u(t-\pi)$

3. Write the function $f(t) = \begin{cases} t^2, & 0 \le t < 1 \\ t^3, & 1 \le t < 2 \\ t, & t \ge 2 \end{cases}$ in terms of the unit step function. (6 points)

$t^2 + (t^3 - t^2)\,u(t-1) + (t - t^3)\,u(t-2)$

4. Use Laplace transforms to solve the IVP $y'' + 4y' + 8y = e^{-t}, y(0) = 0,\ y'(0) = 1$. (10 points)

$$s^2Y(s) - s(0) - 1 + 4sY(s) - 0 + 8Y(s) = \frac{1}{s+1}$$

$$Y(s)(s^2+4s+8) = \frac{1}{s+1} + 1$$

$$Y(s) = \frac{1}{(s+1)(s^2+4s+8)} + \frac{s+1}{(s+1)(s^2+4s+8)} = \frac{s+2}{(s+1)(s^2+4s+8)}$$

$$\frac{A}{s+1} + \frac{Bs+C}{s^2+4s+8} = As^2+4As+8A+Bs^2+Bs+Cs+C$$

$A+B=0$

$4A+B+C=1$

$8A+C=2$ back page

5. Write $x^4 + 3x$ as a power series in terms of $x + 1$. (6 points)

$f(x) = x^4+3x$ $\quad f(-1) =$

$f'(x) = 4x^3+3$ $\quad f'(-1) = -1$

$f''(x) = 12x^2$ $\quad f''(-1) = 12$

$f'''(x) = 24x$ $\quad f'''(-1) = -24$

$f^{iv}(x) = 24$ $\quad f^{iv}(-1) = 24$

$f^{v}(x) = 0$ $\quad f^{v}(-1) = 0$

$$-2 - (x+1) + \frac{12(x+1)^2}{2!} - \frac{24(x+1)^3}{3!} + \frac{24(x+1)^4}{4!}$$

$$-2 - (x+1) + 6(x+1)^2 - 4(x+1)^3 + (x+1)^4$$

6. Rewrite the power series $\sum_{n=3}^{\infty} a_n x^{n-2}$ so that the index starts at $n = 0$. (4 points)

$$\sum_{n=0}^{\infty} a_{n+3}x^{n+1}$$

7. Write $\sum_{n=2}^{\infty} a_n n(n-1)x^{n-2} + x\sum_{n=1}^{\infty} 4a_n x^{n-1}$ as a single sum with x^n. (5 points)

$$\sum_{n=0}^{\infty} a_{n+2}(n+2)(n+1)x^n + \sum_{n=1}^{\infty} 4a_n x^n$$

$$a_2(2)(1)x^0 + \sum_{n=1}^{\infty} a_{n+2}(n+2)(n+1)x^n + \sum_{n=1}^{\infty} 4a_n x^n$$

$$2a_2 + \sum_{n=1}^{\infty}\left[a_{n+2}(n+2)(n+1) + 4a_n\right]x^n = 0$$

8. For each equation, identify any singular points and classify them as regular or irregular. (5 points each)

a. $\frac{x^3-1}{x^2+1}y'' + (x^2-1)y' + y = x$

$$y'' + \frac{(x^2-1)(x^2+1)}{x^3-1}y' + \frac{x^2+1}{x^3-1}y = \frac{x(x^2+1)}{x^3-1}$$

regular at $x=1$

$$y'' + \frac{(x+1)(x^2+1)}{x^2+x+1}y' + \frac{(x^2+1)}{(x-1)(x^2+x+1)}y = \frac{x(x^2+1)}{x^3-1}$$

never 0

0 at $x=1$

Singular at $x=1$

$$\lim_{x\to1}\frac{(x^2+1)(x-1)^{\cancel{2}}}{(\cancel{x-1})(x^2+x+1)} = \text{defined}$$

b. $(x^2+6x)y'' + (3x+9)y' - 3y = 0$

$$y'' + \frac{3(x+3)}{x(x+6)}y' - \frac{3y}{x(x+6)} = 0$$

singular at $x=0$, $x=-6$

$$\lim_{x\to0}\frac{3(x+3)}{\cancel{x}(x+6)}\cdot\cancel{x} = \text{defined} \qquad \lim_{x\to0}\frac{3(x+3)\,\cancel{x^2}}{\cancel{x}(x+6)} = \text{defined}$$

$$\lim_{x\to-6}\frac{3(x+3)\cancel{(x+6)}}{x\cancel{(x+6)}} = \text{defined} \qquad \lim_{x\to-6}\frac{3(x+3)(x+6)^{\cancel{2}}}{x\cancel{(x+6)}} = \text{defined}$$

both are regular

c. $xy'' + (\sin x)y' + xy = 0$

$$y'' + \frac{\sin x}{x}y' + y = 0$$

$x=0$ singular

$$\lim_{x\to0}\frac{\sin x}{\cancel{x}}\cdot\cancel{x} = 0 \text{ defined}$$

regular

9. Use series solution methods to find the solution to $y'' - 2y' + y = 0$. State at least 4 terms of each solution (unless it is finite). Be sure that you find two solutions. (13 points)

$$y = \sum_{n=0}^{\infty} a_n x^n \quad y' = \sum_{n=1}^{\infty} a_n n x^{n-1} \quad y'' = \sum_{n=2}^{\infty} a_n n(n-1) x^{n-2}$$

$$\sum_{n=2}^{\infty} a_n n(n-1) x^{n-2} - 2\sum_{n=1}^{\infty} a_n n x^{n-1} + \sum_{n=0}^{\infty} a_n x^n =$$

$$\sum_{n=0}^{\infty} a_{n+2}(n+2)(n+1)x^n - \sum_{n=0}^{\infty} 2a_{n+1}(n+1)x^n + \sum_{n=0}^{\infty} a_n x^n$$

$$\sum_{n=0}^{\infty} \left[a_{n+2}(n+2)(n+1) - 2a_{n+1}(n+1) + a_n\right]x^n = 0$$

$$a_{n+2} = \frac{2a_{n+1}\cancel{(n+1)} - a_n}{(n+2)\cancel{(n+1)}\;(n+2)(n+1)}$$

$$a_{n+2} = \frac{2}{n+2}a_{n+1} - \frac{1}{(n+2)(n+1)}a_n$$

$n=0$

$$a_2 = a_1 - \frac{1}{2}a_0$$

$n=1$

$$a_3 = \frac{2}{3}a_2 - \frac{1}{6}a_1 = \frac{2}{3}\left(a_1 - \frac{1}{2}a_0\right) - \frac{1}{6}a_1 = \frac{2}{3}a_1 - \frac{1}{6}a_1 \cdot \frac{1}{3}a_0 \cdot$$

$$\frac{1}{2}a_1 - \frac{1}{3}a_0$$

$n=2$

$$a_4 = \frac{2}{4}a_3 - \frac{1}{12}a_2 = \frac{1}{2}\left(\frac{1}{2}a_1 - \frac{1}{3}a_0\right) - \frac{1}{12}\left(a_1 - \frac{1}{2}a_0\right) = \frac{1}{6}a_1 - \frac{1}{8}a_0$$

$n=3$

$$a_5 = \frac{2}{5}a_4 - \frac{1}{20}a_3 = \frac{2}{5}\left(\frac{1}{6}a_1 - \frac{1}{8}a_0\right) - \frac{1}{20}\left(\frac{1}{2}a_1 - \frac{1}{3}a_0\right) = \frac{1}{24}a_1 - \frac{1}{30}a_0$$

$$y(x) = a_0\left(1 - \frac{1}{2}x^2 - \frac{1}{3}x^3 - \frac{1}{8}x^4 - \frac{1}{30}x^5 \ldots\right) + a_1\left(x + x^2 + \frac{1}{2}x^3 + \frac{1}{6}x^4 + \frac{1}{24}x^5 + \cdots\right)$$

10. Solve $y'' + x^2y' + 2xy = 0$ using series methods. State at least 4 terms of each solution (unless it is finite). Be sure that you find two solutions. (15 points)

$$y=\sum_{n=0}^{\infty} a_n x^n \quad y'=\sum_{n=1}^{\infty} a_n n x^{n-1} \quad y''=\sum_{n=2}^{\infty} a_n n(n-1)x^{n-2}$$

$$\sum_{n=2}^{\infty} a_n n(n-1)x^{n-2} + x^2\sum_{n=1}^{\infty} a_n n x^{n-1} + 2x\sum_{n=0}^{\infty} a_n x^n =$$

$$\sum_{n=2}^{\infty} a_n n(n-1)x^{n-2} + \sum_{n=1}^{\infty} a_n n x^{n+1} + \sum_{n=0}^{\infty} 2a_n x^{n+1} =$$

$$\sum_{n=-1}^{\infty} a_{n+3}(n+3)(n+2)x^{n+1} + \sum_{n=1}^{\infty} a_n n x^{n+1} + \sum_{n=0}^{\infty} 2a_n x^{n+1}$$

$$a_2(2)(1)(1) + a_3(3)(2)x + \sum_{n=1}^{\infty}\left[a_{n+3}(n+3)(n+2) + a_n n + 2a_n\right]x^{n+1} = 0$$

$a_n(n+2)$

$2a_2=0 \quad 6a_3=0$

$$a_{n+3} = -\frac{a_n\cancel{(n+2)}}{(n+3)\cancel{(n+2)}} = \frac{-a_n}{n+3}$$

$a_2=0,\ a_3=0$

$n=1 \quad a_4 = \frac{-a_1}{4}$ $\quad n=2 \quad a_5 = \frac{-a_2}{5} = 0$

$n=3 \quad a_6 = \frac{-a_3}{6} = 0$ $\quad n=4 \quad a_7 = \frac{-a_4}{7} = \frac{1}{28}a_1$

$n=5 \quad a_8 = \frac{-a_5}{8} = 0$ $\quad n=6 \quad a_9 = \frac{-a_6}{9} = 0$

$n=7 \quad a_{10} = \frac{-a_7}{10} = \frac{-1}{280}a_1$

$$y(x) = a_0 + a_1\left(x - \tfrac{1}{4}x^4 + \tfrac{1}{28}x^7 - \tfrac{1}{280}x^{10} \ldots\right)$$

11. Use the method of Frobenius to find the solution to the differential equation $2xy'' - y' - y = 0$. Stop when you obtain values for r. (11 points)

$$y=\sum_{n=0}^{\infty} a_n x^{n+r} \quad y'=\sum_{n=0}^{\infty} a_n(n+r)x^{n+r-1} \quad y''=\sum_{n=0}^{\infty} a_n(n+r)(n+r-1)x^{n+r-2}$$

$$2x\sum_{n=0}^{\infty} a_n(n+r)(n+r-1)x^{n+r-2} - \sum_{n=0}^{\infty} a_n(n+r)x^{n+r-1} - \sum_{n=0}^{\infty} a_n x^{n+r} =$$

$$\sum_{n=0}^{\infty} 2a_n(n+r)(n+r-1)x^{n+r-1} - \sum_{n=0}^{\infty} a_n(n+r)x^{n+r-1} - \sum_{n=0}^{\infty} a_n x^{n+r} = 0$$

$$\sum_{n=-1}^{\infty} 2a_{n+1}(n+r+1)(n+r)x^{n+r} - \sum_{n=-1}^{\infty} a_{n+1}(n+r+1)x^{n+r} - \sum_{n=0}^{\infty} a_n x^{n+r}$$

$$2(a_0)(r)(r-1)x^{r-1} - a_0(r)x^{r-1} + \sum_{n=0}^{\infty}\left[2a_{n+1}(n+r+1)(n+r) - a_{n+1}(n+r+1) - a_n\right]x^{n+r} = 0$$

$$a_0x^{r-1}\left[2r(r-1) - r\right] = 0$$

$2r^2 - 2r - r = 2r^2 - 3r = 0$

$r=0 \quad r=\frac{3}{2}$

last page

4. cont'd

$A = 1/5 \quad B = -1/5 \quad C = 2/5$

$\frac{1}{5}\left(\frac{1}{s+1}\right) + \frac{1}{5}\left(\frac{s+2}{s^2+4s+8}\right)$ $\qquad s^2+4s+4+4$

$\qquad (s+2)^2+4$

$\frac{1}{5}\left(\frac{1}{s+1}\right) + \frac{1}{5}\left(\frac{s+2}{(s+2)^2+4}\right) = Y(s)$

$y(t) = \frac{1}{5}e^{-t} + \frac{1}{5}e^{-2t}\cos 2t$

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11. cont'd

$r=0$

$$\sum_{n=0}^{\infty}\left[2a_{n+1}(n+1)n - a_{n+1}(n+1) - a_n\right] = 0$$

$a_{n+1}\left[2(n^2+n) - (n+1)\right] = a_n$

$2n^2+2n-n-1$

$2n^2+n-1$

$(2n-1)(n+1)$

$a_{n+1} = \frac{a_n}{(2n-1)(n+1)}$

$n=0 \quad a_1 = \frac{a_0}{(-1)(1)} = -a_0 \qquad n=2 \quad a_3 = \frac{a_2}{3(3)} = \frac{-a_0}{2\cdot 9} = -\frac{a_0}{18}$

$n=1 \quad a_2 = \frac{a_1}{(1)(2)} = \frac{-a_0}{2} \qquad n=3 \quad a_4 = \frac{a_3}{5(4)} = -\frac{a_0}{18}\left(\frac{1}{20}\right) = \frac{-a_0}{360}$

$r = \frac{3}{2} \quad \sum_{n=0}^{\infty}\left[2a_{n+1}\left(n+\frac{5}{2}\right)\left(n+\frac{3}{2}\right) - a_{n+1}\left(n+\frac{5}{2}\right) - a_n\right] = 0$

$2\left(n^2 + \frac{3}{2}n + \frac{5}{2}n + \frac{15}{4}\right) - n - \frac{5}{2}$

$2n^2 + \;n + \frac{15}{2} - n - \frac{5}{2} = 2n^2 + 7n + 5$

$a_{n+1} = \frac{a_n}{(2n+5)(n+1)} \qquad n=0 \quad \frac{a_0}{5(1)} = a_1 = \frac{a_0}{5}$

$n=1 \quad \frac{a_1}{7(2)} = a_2 = \frac{a_0}{70} \qquad n=2 \quad a_3 = \frac{a_2}{9(3)} = \frac{a_0}{1890}$

$$y(x) = a_0\left(1 - x - \frac{1}{2}x^2 - \frac{1}{18}x^3 - \frac{1}{360}x^4 \dots\right) + a_1 x^{3/2}\left(1 + \frac{1}{5}x - \frac{1}{70}x^2 + \frac{1}{1890}x^3 + \dots\right)$$