MAT 212, Final Exam, Spring 2017 Name KEY

**Instructions**: Show all work. Answers without work required to obtain the solution will not receive full credit. Some questions may contain multiple parts: be sure to answer all of them. Give exact answers unless specifically asked to estimate.

1. Estimate the solution of the ODE $\frac{dy}{dx} = y\cos x, y(0) = 1$ using $\Delta t = 0.1$ using two complete steps of Runge-Kutta. (12 points)

$x_0 = 0 \quad y_0 = 1 \qquad k_{01} = (1)\cos(0) = 1 \qquad k_{02} = (1 + .05(1))\cos(.05) = 1.049$

$k_{03} = (1 + .05(1.049))\cos(.05) = 1.0511$

$k_{04} = (1 + .1(1.0511))\cos(.1) = 1.09959$

$x_1 = 0.1 \quad y_1 = \frac{1}{6}\left(1 + 2(1.049) + 2(1.0511) + 1.09959\right)\cdot 0.1 + 1 = 1.1049965$

$k_{11} = 1.105(\cos(.1)) = 1.099476$

$k_{12} = (1.099476 * .05 + 1.105)\cos(.15) = 1.1469$

$k_{13} = (1.1469 * .05 + 1.105)\cos(.15) = 1.14929$

$k_{14} = (1.14929 * .1 + 1.105)\cos(.2) = 1.1956$

$x_2 = 0.2 \quad y_2 = \frac{1}{6}(.1)\left(1.1956 + 2(1.14929) + 2(1.1469) + 1.099476\right) + 1.105$

$= 1.21979$

2. Three tanks (A, B, and C) are coupled together. Suppose that tank A has 500L of water containing 10 kg of salt, that tank B has 400L of pure water, and that tank C has 1000L of water with 100kg of salt. Pure water flows into tank A at a rate of 5L/s; the well-mixed solution flows into tank B at the same rate, and tank B flows into tank C at the same rate. The well-mixed solution flows out of tank C at the same rate. Write the system that models the amount of salt in each tank at time $t$. [You don't need to solve, just set it up.] (10 points)

$\frac{dA}{dt} = -\frac{A}{500}\cdot 5$

$\frac{dB}{dt} = \frac{A}{100} - \frac{B}{400}\cdot 5 \quad \Rightarrow \quad \begin{bmatrix} A \\ B \\ C \end{bmatrix}' = \begin{bmatrix} -\frac{1}{100} & 0 & 0 \\ \frac{1}{100} & -\frac{1}{80} & 0 \\ 0 & \frac{1}{80} & -\frac{1}{200} \end{bmatrix} \begin{bmatrix} A \\ B \\ C \end{bmatrix}$

$\frac{dC}{dt} = \frac{B}{80} - \frac{C}{1000}\cdot 5$

$A(0) = 10$

$B(0) = 0$

$C(0) = 100$

3. Write the system $\begin{cases} x_1' = x_2 \\ x_2' = 2x_3 \\ x_3' = 3x_4 \\ x_4' = 4x_1 \end{cases}$ as a single fourth-order equation. (8 points)

$x_2 = x_1'$

$x_2' = x_1''$

$x_2' = 2x_3$

$\frac{1}{2}x_1'' = x_3$

$\frac{1}{2}x_1''' = x_3'$

$\frac{1}{2}x_1''' = 3x_4$

$\frac{1}{6}x_1''' = x_4$

$\frac{1}{6}x_1^{IV} = x_4'$

$x_4' = 4x_1$

$\frac{1}{6}x_1^{IV} = 4x_1$

$x_1^{IV} = 24x_1$

$\boxed{x_1^{IV} - 24x_1 = 0}$

4. Rewrite $y^{IV} + 2y''' + y' + y = \cos 2t - 6\sin 2t$ as a system of first order equations. (You don't need to solve.) (8 points)

$y = x_1$

$y' = x_2 = x_1'$

$y'' = x_3 = x_2'$

$y''' = x_4 = x_3'$

$y^{IV} = x_4'$

$x_1' = x_2$

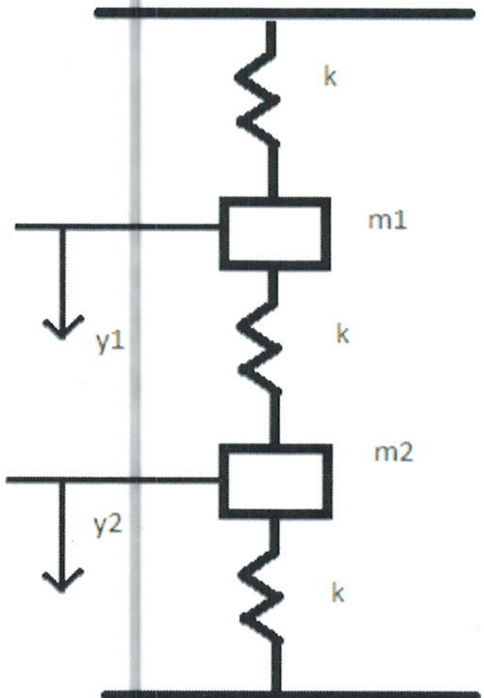
$x_2' = x_3$

$x_3' = x_4$

$x_4' = -2x_4 - x_2 - x_1$

$$\vec{X}' = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & -2 & 0 & -2 \end{bmatrix} \vec{X} + \begin{bmatrix} 0 \\ 0 \\ 0 \\ \cos 2t - 6\sin 2t \end{bmatrix}$$

5. Set up the spring problem shown below as a system of equations. (You don't need to solve.) Let $m_1 = 3, m_2 = 1, k = 5$. (10 points)




$$my_1'' = -2ky_1 - k \quad y_2$$
$$my_2'' = -2ky_2 - ky_1$$

$$3y_1'' = -10y_1 - 5y_2$$
$$y_2'' = -5y_1 - 10y^2$$

$$\begin{bmatrix} y_1 \\ y_2 \end{bmatrix}'' = \begin{bmatrix} -10/3 & -5/3 \\ -5 & -10 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$$

6. Solve $\vec{x}' = \begin{pmatrix} 1 & 3 \\ -1 & -2 \end{pmatrix} \vec{x}$. Write the general solution (with real terms only). Plot several sample trajectories. (10 points)

$$\vec{x} = c_1 e^{-1/2 t} \begin{pmatrix} -3\cos\left(\frac{\sqrt{3}}{2}t\right) + \sqrt{3}\sin\left(\frac{\sqrt{3}}{2}t\right) \\ 2\cos\left(\frac{\sqrt{3}}{2}t\right) \end{pmatrix} + c_2 e^{-1/2 t} \begin{pmatrix} -3\sin\left(\frac{\sqrt{3}}{2}t\right) - \sqrt{3}\cos\left(\frac{\sqrt{3}}{2}t\right) \\ 2\sin\left(\frac{\sqrt{3}}{2}t\right) \end{pmatrix}$$

$$(1-\lambda)(-2-\lambda) + 3 = 0$$
$$\lambda^2 - \lambda + 2\lambda - 2 + 3 = 0$$
$$\lambda^2 + \lambda + 1 = 0$$
$$\lambda = \frac{-1 \pm \sqrt{1-4}}{2}$$
$$\lambda = \frac{-1 \pm \sqrt{3}i}{2}$$

Spirals in

$\lambda_1 =$

$$\begin{bmatrix} 1 - \left(\frac{-1}{2} + \frac{\sqrt{3}}{2}i\right) & 3 \\ -1 & -2 - \left(-\frac{1}{2} + \frac{\sqrt{3}}{2}i\right) \end{bmatrix}$$

$$\vec{v}_1 = \begin{bmatrix} -3 - \sqrt{3}i \\ 2 \end{bmatrix} \qquad \vec{v}_2 = \begin{bmatrix} -3 + \sqrt{3}i \\ 2 \end{bmatrix}$$

$$e^{-1/2 t}\begin{bmatrix} -3 - \sqrt{3}i \\ 2 \end{bmatrix}\left[\cos\left(\frac{\sqrt{3}}{2}t\right) + i\sin\left(\frac{\sqrt{3}}{2}t\right)\right]$$

$$= e^{-1/2 t}\begin{pmatrix} -3\cos\left(\frac{\sqrt{3}}{2}t\right) - 3i\sin\left(\frac{\sqrt{3}}{2}t\right) - \sqrt{3}i\cos\left(\frac{\sqrt{3}}{2}t\right) + \sqrt{3}\sin\left(\frac{\sqrt{3}}{2}t\right) \\ 2\cos\left(\frac{\sqrt{3}}{2}t\right) + i2\sin\left(\frac{\sqrt{3}}{2}t\right) \end{pmatrix}$$

$$\begin{bmatrix} \frac{3}{2} - \frac{\sqrt{3}}{2}i & 3 \\ -1 & -\frac{3}{2} - \frac{\sqrt{3}}{2}i \end{bmatrix}$$

$$x_1 = \left(-\frac{3}{2} - \frac{\sqrt{3}}{2}i\right)x_2$$
$$x_2 = x_2$$

7. Verify that $\Psi = \begin{pmatrix} 3e^{-2t} & e^{t} & e^{3t} \\ -2e^{-2t} & -e^{t} & -e^{3t} \\ 2e^{-2t} & e^{t} & 0 \end{pmatrix}$ is a solution to $\vec{x}' = \begin{pmatrix} -8 & -11 & -2 \\ 6 & 9 & 2 \\ -6 & -6 & 1 \end{pmatrix}\vec{x}$. Does the solution represent a fundamental set? (8 points)

$$\Psi' = \begin{pmatrix} -6e^{-2t} & e^{t} & 3e^{3t} \\ 4e^{-2t} & -e^{t} & -3e^{3t} \\ -4e^{-2t} & e^{t} & 0 \end{pmatrix}$$

$$\begin{pmatrix} -8 & -11 & -2 \\ 6 & 9 & 2 \\ -6 & -6 & 1 \end{pmatrix}\begin{pmatrix} 3e^{-2t} & e^{t} & e^{3t} \\ -2e^{-2t} & -e^{t} & -e^{3t} \\ 2e^{-2t} & e^{t} & 0 \end{pmatrix} =$$

$$\begin{pmatrix} -24e^{-2t}+22e^{-2t}-4e^{-2t} & -8e^{t}+11e^{t}-2e^{t} & -8e^{3t}+11e^{3t}+0 \\ 18e^{-2t}-18e^{2t}+4e^{2t} & 6e^{t}-9e^{t}+2e^{t} & 6e^{3t}-9e^{3t}+0 \\ -18e^{-2t}+12e^{2t}+2e^{-2t} & -6e^{t}+6e^{t}+e^{t} & -6e^{3t}+6e^{3t}+0 \end{pmatrix}$$

$$= \begin{pmatrix} -6e^{-2t} & e^{t} & 3e^{3t} \\ 4e^{2t} & -e^{t} & -3e^{3t} \\ -4e^{-2t} & e^{t} & 0 \end{pmatrix}$$

8. Solve $\vec{x}' = \begin{bmatrix} 1 & -3 \\ 3 & 7 \end{bmatrix}\vec{x}$ for the general solution (with real terms only). Describe the behavior of the origin: a repeller, attracter, or saddle point.(8 points)

$(1-\lambda)(7-\lambda)+9$

$\lambda^2 - 8\lambda + 7 + 9 =$

$\lambda^2 - 8\lambda + 16 = 0$

$(\lambda-4)^2 = 0$

$\lambda = 4$

$\begin{bmatrix} 1-4 & -3 \\ 3 & 7-4 \end{bmatrix}$

$\begin{bmatrix} -3 & -3 \\ 3 & 3 \end{bmatrix}$

$x_1 = -x_2$
$x_2 = x_2$ $= \begin{bmatrix} -1 \\ 1 \end{bmatrix}$

$\left[\begin{array}{cc|c} -3 & -3 & -1 \\ 3 & 3 & 1 \end{array}\right] \Rightarrow \left[\begin{array}{ccc} 1 & 1 & 1/3 \\ 0 & 0 & 0 \end{array}\right]$

$x_1 = -x_2 + 1/3$

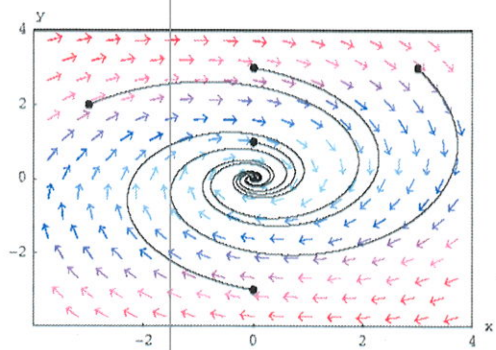
$x_2 = x_2 + 0$

$\vec{x} = c_1\begin{bmatrix} -1 \\ 1 \end{bmatrix}e^{4t} + c_2\left(\begin{bmatrix} -1 \\ 1 \end{bmatrix}te^{4t} + \begin{bmatrix} \frac{1}{3} \\ 0 \end{bmatrix}e^{4t}\right]$

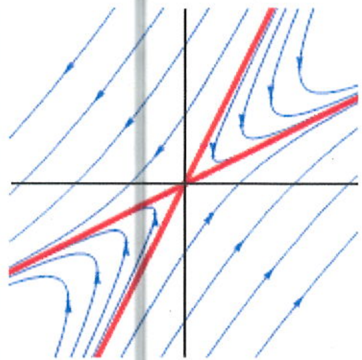
Origin is a repeller

9. For each set of solution curves shown below, match the graphs with proposed solutions and characterize the system as containing a stable vector/orbit, origin attracts, origin repels, origin is a saddle point. (12 points)

ii



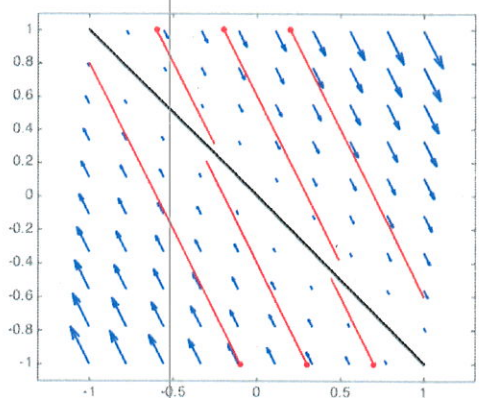
a.



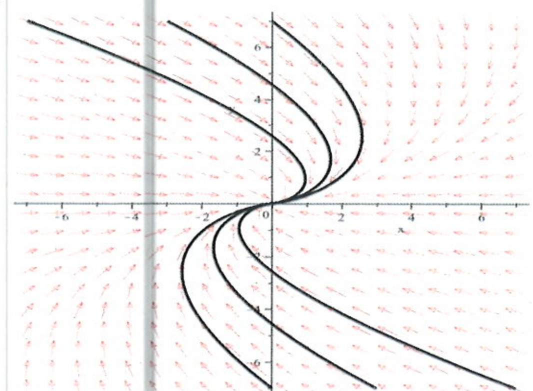
d.

iii

iv



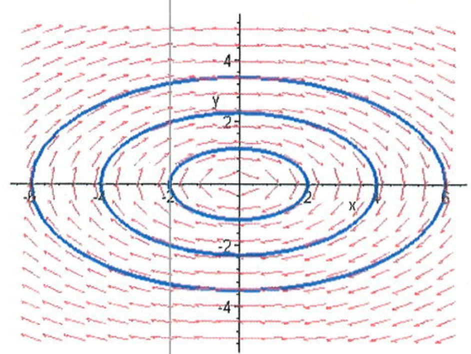
b.



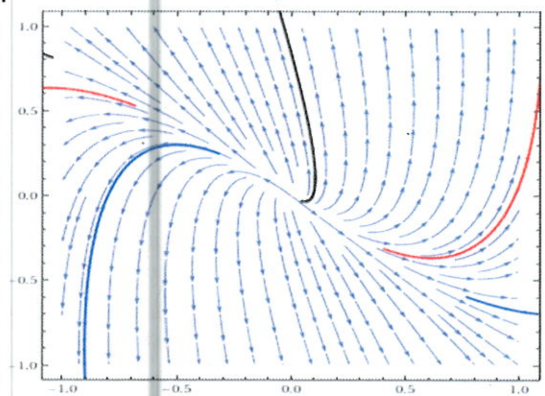
e.

vi

i



c.



f.

v

i. $\vec{x}(t) = c_1 \begin{pmatrix} 2\cos t \\ \sin t \end{pmatrix} + c_2 \begin{pmatrix} -\sin t \\ \cos t \end{pmatrix}$ C

ii. $\vec{x}(t) = c_1 e^{-t} \begin{pmatrix} 3\cos t + \sin t \\ 2\sin t \end{pmatrix} + c_2 e^{-t} \begin{pmatrix} -3\sin t + cost \\ -2\cos t \end{pmatrix}$ A

iii. $\vec{x}(t) = c_1 \begin{pmatrix} 2 \\ 1 \end{pmatrix} e^{-t} + c_2 \begin{pmatrix} 1 \\ 2 \end{pmatrix} e^{2t}$ D

iv. $\vec{x}(t) = c_1 \begin{pmatrix} -1 \\ 1 \end{pmatrix} + c_2 \begin{pmatrix} 1 \\ 4 \end{pmatrix} e^{-2t}$ B

v. $\vec{x}(t) = c_1 \begin{pmatrix} -1 \\ 1 \end{pmatrix} e^{3t} + c_2 \begin{pmatrix} 3 \\ 2 \end{pmatrix} e^{2t}$ F

vi. $\vec{x}(t) = c_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{-t} + c_2 \begin{pmatrix} 1 \\ -2 \end{pmatrix} e^{-3t}$ E

10. The fundamental solution matrix to the system $\vec{x}' = \begin{pmatrix} 2 & 4 \\ 2 & -5 \end{pmatrix}\vec{x}$ is $\Psi = \begin{pmatrix} -e^{-6t} & -4e^{3t} \\ 2e^{-6t} & e^{3t} \end{pmatrix}$. Use this fact to solve the system $\vec{x}' = \begin{pmatrix} 2 & 4 \\ 2 & -5 \end{pmatrix}\vec{x} + \begin{pmatrix} 3e^{t} \\ -t^2 \end{pmatrix}$. Recall that $\vec{x}_p = \Psi \int \Psi^{-1} f(t)dt$ or use the method of undetermined coefficients. (15 points)

$$W = \begin{vmatrix} -e^{-6t} & -4e^{3t} \\ 2e^{-6t} & e^{3t} \end{vmatrix} = -e^{-3t} + 8e^{-3t} = 7e^{-3t}$$

$$\frac{e^{3t}}{7}\begin{bmatrix} e^{3t} & 4e^{3t} \\ -2e^{-6t} & -e^{-6t} \end{bmatrix} = \begin{bmatrix} e^{6t}/7 & 4/7\, e^{6t} \\ -\frac{2}{7}e^{-3t} & -\frac{1}{7}e^{-3t} \end{bmatrix}$$

$$\begin{bmatrix} \frac{1}{7}e^{6t} & \frac{4}{7}e^{6t} \\ -\frac{2}{7}e^{-3t} & -\frac{1}{7}e^{-3t} \end{bmatrix}\begin{bmatrix} 3e^{t} \\ -t^2 \end{bmatrix} = \begin{bmatrix} \frac{3}{7}e^{7t} - \frac{4}{7}e^{6t}\cdot t^2 \\ -\frac{6}{7}e^{-2t} + \frac{1}{7}t^2e^{-3t} \end{bmatrix}$$

| $\pm$ | $u$ | $dv$ |
|---|---|---|
| + | $\frac{4}{7}t^2$ | $e^{6t}$ |
| − | $\frac{8}{7}t$ | $\frac{1}{6}e^{6t}$ |
| + | $8/7$ | $\frac{1}{36}e^{6t}$ |
| − | 0 | $\frac{1}{216}e^{6t}$ |

$$\int \frac{3}{7}e^{7t} - \frac{4}{7}t^2e^{6t}\,dt = \frac{3}{49}e^{7t} - \frac{2}{21}t^2e^{6t} + \frac{2}{63}te^{6t} - \frac{1}{189}e^{6t}$$

$$\int -\frac{6}{7}e^{-2t} + \frac{1}{7}t^2e^{-3t}\,dt = \frac{3}{7}e^{-2t} + -\frac{1}{21}t^2e^{-3t} - \frac{2}{63}te^{-3t} - \frac{2}{189}e^{-3t}$$

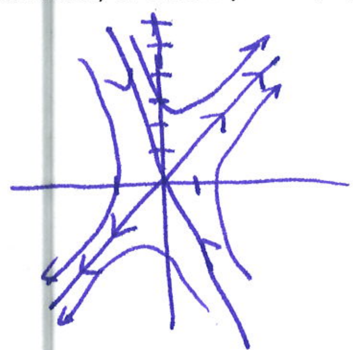
$$\begin{bmatrix} -e^{-6t} & -4e^{3t} \\ 2e^{-6t} & e^{3t} \end{bmatrix}\begin{bmatrix} \frac{3}{49}e^{7t} + e^{6t}\left(-\frac{2}{21}t^2 + \frac{2}{63}t - \frac{1}{189}\right) \\ \frac{3}{7}e^{-2t} + e^{-3t}\left(-\frac{1}{21}t^2 - \frac{2}{63}t - \frac{2}{189}\right) \end{bmatrix}$$

| $\pm$ | $u$ | $dv$ |
|---|---|---|
| + | $\frac{1}{7}t^2$ | $e^{-3t}$ |
| − | $\frac{2}{7}t$ | $-\frac{1}{3}e^{-3t}$ |
| + | $\frac{2}{7}$ | $+\frac{1}{9}e^{-3t}$ |
| − | 0 | $-\frac{1}{27}e^{-3t}$ |

$$= \begin{bmatrix} -\frac{3}{49}e^{t} - \left(-\frac{2}{21}t^2 + \frac{2}{63}t - \frac{1}{189}\right) - \frac{12}{7}e^{t} + \left(\frac{4}{21}t^2 + \frac{8}{63}t + \frac{8}{189}\right) \\ \frac{6}{49}e^{t} + \left(-\frac{4}{21}t^2 + \frac{4}{63}t - \frac{2}{189}\right) + \frac{3}{7}e^{t} + \left(-\frac{1}{21}t^2 - \frac{2}{63} - \frac{2}{189}\right) \end{bmatrix}$$

$$= \begin{bmatrix} -87/49\, e^{t} + \frac{6}{21}t^2 + \frac{6}{63}t + \frac{9}{189} \\ [illegible]/49\, e^{t} - \frac{5}{21}t^2 + \frac{2}{63}t - \frac{4}{189} \end{bmatrix}$$

11. Find the eigenvalues and eigenvectors of the system $\vec{x}' = \begin{bmatrix} 4 & 1 \\ 6 & -1 \end{bmatrix}\vec{x}$. Draw the phase plane and plot several sample trajectories. Is the origin a repeller, attracter, or saddle point. (8 points)



$(4-\lambda)(-1-\lambda) - 6 = 0$

$\lambda^2 - 3\lambda - 4 - 6 = 0$

$\lambda^2 - 3\lambda - 10 = 0$

$(\lambda - 5)(\lambda + 2) = 0$

$\lambda = 5,\ \lambda = -2$

saddle point

$\lambda = 5$

$$\begin{bmatrix} -1 & 1 \\ 6 & -6 \end{bmatrix} \quad \begin{aligned} x_1 &= x_2 \\ x_2 &= x_2 \end{aligned}$$

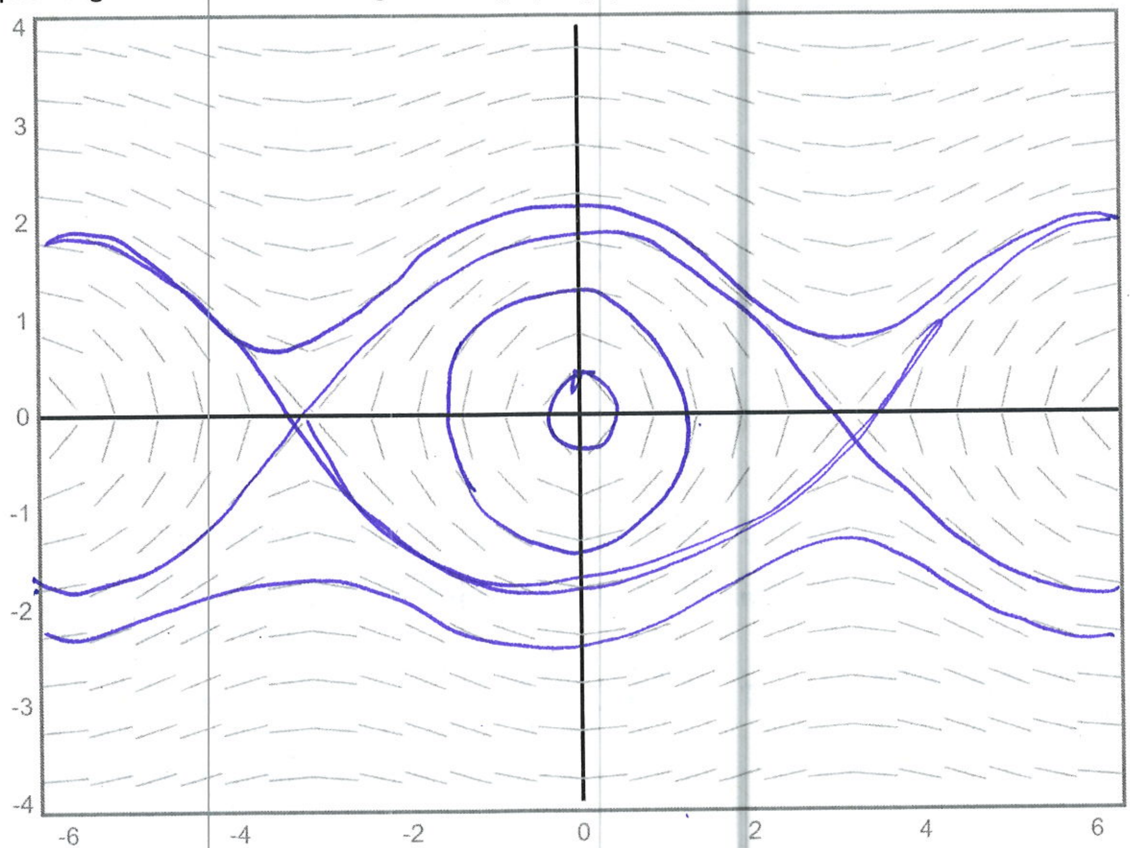
$$\vec{v}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$\lambda = -2$

$$\begin{bmatrix} 6 & 1 \\ 6 & 1 \end{bmatrix} \quad \begin{aligned} x_1 &= -\tfrac{1}{6}x_2 \\ x_2 &= x_2 \end{aligned}$$

$$\vec{v}_2 = \begin{bmatrix} -1 \\ 6 \end{bmatrix}$$

12. Shown below is the slope field for an undamped pendulum. Plot at least three sample trajectories with different behaviors. Track the path forward and backward in time (so that the path begins and ends on the edges of the graph. (6 points)




13. Solve $x^2y' = 1 - x^2 + y^2 - x^2y^2$ by separation of variables. [Hint: Factor by grouping.] (12 points)

$$x^2y' = (1-x^2) + y^2(1-x^2)$$

$$x^2y' = (1+y^2)(1-x^2)$$

$$\frac{dy}{1+y^2} = \frac{1-x^2}{x^2}\,dx = (x^{-2}-1)\,dx$$

$$\arctan y = -\frac{1}{x} - x + C$$

18. Use the table of Laplace transforms to find Laplace transforms or inverse Laplace transforms as indicated. (4 points each)

a. $\mathcal{L}\{(1+2t)^2\}$

$1+4t+4t^2$

$\frac{1}{s}+\frac{4}{s^2}+\frac{8}{s^3}$

b. $\mathcal{L}\{e^{-2t}\sin 3t\}$

$\frac{3}{(s+2)^2+9}$

c. $\mathcal{L}\left\{\frac{\sin t}{t}\right\}$

$\arctan\left(\frac{1}{s}\right)$

d. $\mathcal{L}\left\{\frac{1}{2}\int_0^t (t-\tau)^3 \sin 2\tau \, d\tau\right\}$

$\frac{1}{\cancel{2}}\cdot\frac{6}{s^4}\cdot\frac{\cancel{2}}{s^2+4} = \frac{6}{s^4(s^2+4)}$

e. $\mathcal{L}^{-1}\left\{\frac{1}{2}-\frac{2}{s^5}\right\}$

$\frac{1}{2}\delta(t) - \frac{1}{12}t^4$

f. $\mathcal{L}^{-1}\left\{\frac{9-17s}{s^2+81}\right\}$

$\frac{9}{s^2+81} - \frac{17s}{s^2+18}$

$\sin 9t - 17\cos 9t$

g. $\mathcal{L}^{-1}\left\{\frac{1}{s^2(s^2-1)}\right\}$

$\frac{1}{s^2} \cdot \frac{1}{s^2-1}$

$\int_0^t (t-\tau) \sinh \tau \, d\tau$

h. $\mathcal{L}^{-1}\left\{\frac{e^{-\pi s}}{s^2+1}\right\}$ $e^{-\pi s}\left(\frac{1}{s^2+1}\right)$

$\sin t$

$\sin(t-\pi)\, u(t-\pi)$

19. Use Laplace transforms to solve the IVP $y'' + 4y' + 8y = e^{-t}, y(0) = 0,\ y'(0) = 1$. (10 points)

$s^2 Y(s) - 1 + 4sY(s) + 8Y(s) = \frac{1}{s+1}$

$Y(s)(s^2 + 4s + 8) = \frac{1+s+1}{s+1} = \frac{s+2}{s+1}$

$Y(s) = \frac{s+2}{(s+1)(s^2+4s+8)} = \frac{A}{s+1} + \frac{Bs+C}{s^2+4s+8}$

$As^2 + 4As + 8A + Bs^2 + Bs + Cs + C = s+2$

$A + B = 0$ $A = 1/5$ $1/5\left(\frac{1}{s+1}\right) + \frac{1}{5}\left(\frac{s+2}{(s+2)^2+4}\right)$

$4A + B + C = 1$ $B = -1/5$

$8A + C = 2$ $C = 2/5$

$y(t) = \frac{1}{5}e^{-t} + \frac{1}{5}e^{-2t}\cos 2t$

20. Solve $y'' + x^2y' + 2xy = 0$ using series methods. State at least 4 terms of each solution (unless it is finite). Be sure that you find two solutions. (15 points)

$$y=\sum_{n=0}^{\infty} a_n x^n \quad y'=\sum_{n=1}^{\infty} a_n n x^{n-1} \quad y''=\sum_{n=2}^{\infty} a_n x^n n(n-1)$$

$$\sum_{n=2}^{\infty} a_n n(n-1)x^{n-2} + x^2\sum_{n=1}^{\infty} a_n n x^{n-1} + 2x\sum_{n=0}^{\infty} a_n x^n = 0$$

$$\sum_{n=2}^{\infty} a_n n(n-1)x^{n-2} + \sum_{n=1}^{\infty} a_n n x^{n+1} + \sum_{n=0}^{\infty} 2a_n x^{n+1} =$$

$$\sum_{n=-1}^{\infty} a_{n+3}(n+3)(n+2)x^{n+1} + \sum_{n=1}^{\infty} a_n n x^{n+1} + \sum_{n=0}^{\infty} 2a_n x^{n+1}$$

$$a_2(2)(1) + a_3(3)(2)x + \sum_{n=1}^{\infty} a_{n+3}(n+3)(n+2)x^{n+1} + \sum_{n=1}^{\infty} a_n n x^{n+1} +$$
$$2a_0x + \sum_{n=1}^{\infty} 2a_n x^{n+1}$$

$2a_2 = 0 \Rightarrow a_2 = 0$

$6a_3 + 2a_0 = 0 \Rightarrow a_3 = -\frac{1}{3}a_0$

$$\sum_{n=1}^{\infty}\Big[\,a_{n+3}(n+3)(n+2) + \underbrace{a_n n + a_n}_{a_n(n+1)}\Big]x^{n+1} = 0$$

$$a_{n+3} = \frac{-a_n(n+1)}{(n+3)(n+2)}$$

$n=1 \quad a_4 = \dfrac{-a_1 2}{4\cdot 3} = -\dfrac{a_1}{6}$

$n=2 \quad a_5 = \dfrac{-a_2\cdot 3}{5\cdot 4} = 0$

$n=3 \quad a_6 = \dfrac{-a_3 4}{6\cdot 5} = \dfrac{-2}{15}\left(\dfrac{-1}{3}a_0\right) = \dfrac{2}{45}a_0$

$n=4 \quad a_7 = \dfrac{-a_4(5)}{7\cdot 6} = \dfrac{-5}{42}\left(\dfrac{-1}{6}a_1\right) = \dfrac{5}{252}a_1$

$n=5 \quad a_8 = 0$

$n=6 \quad a_9 = \dfrac{-a_6(7)}{9\cdot 8} = \left(\dfrac{-7}{72}\right)\left(\dfrac{2}{45}a_0\right) = \dfrac{-7}{1620}a_0$

$n=7 \quad a_{10} = \dfrac{-a_7(8)}{10\cdot 9} = \dfrac{-4}{45}\left(\dfrac{5}{252}\right)a_1 = \dfrac{-1}{567}a_1$

$$y(x) = a_0\left(1 - \frac{1}{3}x^3 + \frac{2}{45}x^6 - \frac{7}{1620}x^9 + \dots\right) + a_1\left(x - \frac{1}{6}x^4 + \frac{5}{252}x^7 - \frac{1}{567}x^{10} + \dots\right)$$

Numerical Solutions Formula Sheet

Euler's $y_{n+1} = y_n + \Delta t\, f(t_n, y_n)$

Error in Euler' $|y_n - y(t_n)| \le C\Delta t$

Improved Euler's

$$k_1 = f(t_n, y_n)$$
$$u_{n+1} = y_n + \Delta t\, k_1$$
$$k_2 = f(t_{n+1}, u_{n+1})$$
$$y_{n+1} = y_n + \frac{1}{2}\Delta t(k_1 + k_2)$$

Error in Improved Euler's $|y(t_n) - y_n| \le C\Delta t^2$

Runge-Kutta

$$y_{n+1} = y_n + h\left(\frac{k_{n1} + 2k_{n2} + 2k_{n3} + k_{n4}}{6}\right)$$
$$k_{n1} = f(t_n, y_n)$$
$$k_{n2} = f\left(t_n + \frac{1}{2}h, y_n + \frac{1}{2}hk_{n1}\right)$$
$$k_{n3} = f\left(t_n + \frac{1}{2}h, y_n + \frac{1}{2}hk_{n2}\right)$$
$$k_{n4} = f(t_n + h, y_n + hk_{n3})$$

Error in Runge-Kutta $|y(t_n) - y_n| \le C\Delta t^4$

# Laplace Transforms Table

*for $t \geq 0$*



| $f(t) = \mathfrak{L}^{-1}\{F(s)\}$ | $F(s) = \mathfrak{L}\{f(t)\}$ | $f(t) = \mathfrak{L}^{-1}\{F(s)\}$ | $F(s) = \mathfrak{L}\{f(t)\}$ |
|---|---|---|---|
| $\mathbf{1}$ | $\frac{1}{s}$ | $e^{at}$ | $\frac{1}{s-a}$ |
| $t^n$ | $\frac{n!}{s^{n+1}}$ | $\sin(at)$ | $\frac{a}{s^2+a^2}$ |
| $t^a$, $a > -1$ | $\frac{\Gamma(a+1)}{s^{a+1}}$ | $\cos(at)$ | $\frac{s}{s^2+a^2}$ |
| $\sqrt{t}$ | $\frac{\sqrt{\pi}}{2s^{3/2}}$ | $t\sin(at)$ | $\frac{2as}{(s^2+a^2)^2}$ |
| $t^{n-\frac{1}{2}}$, $n = 1, 2, 3 \ldots$ | $\frac{1 \cdot 3 \cdot 5 \cdot \ldots \cdot (2n-1)\sqrt{\pi}}{2^n s^{n+\frac{1}{2}}}$ | $t\cos(at)$ | $\frac{s^2-a^2}{(s^2+a^2)^2}$ |
| $f(ct)$ | $\frac{1}{c}F\left(\frac{s}{c}\right)$ | $\sin(at+b)$ | $\frac{s\sin(b)+a\cos(b)}{s^2+a^2}$ |
| $u_c(t) = u(t-c)$ | $\frac{e^{-cs}}{s}$ | $\cos(at+b)$ | $\frac{s\cos(b)-a\sin(b)}{s^2+a^2}$ |
| $\delta(t)$ | $1$ | $\sin(at) - at\cos(at)$ | $\frac{2a^3}{(s^2+a^2)^2}$ |
| $\delta(t-c)$ | $e^{-cs}$ | $\sin(at) + at\cos(at)$ | $\frac{2as^2}{(s^2+a^2)^2}$ |
| $\delta^{(n)}$ | $s^n$ | $\cos(at) - at\sin(at)$ | $\frac{s(s^2-a^2)}{(s^2+a^2)^2}$ |
| $u_c(t)f(t-c)$ | $e^{-cs}F(s)$ | $\cos(at) + at\sin(at)$ | $\frac{s(s^2+3a^2)}{(s^2+a^2)^2}$ |
| $u_c(t)g(t)$ | $e^{-cs}\mathfrak{L}\{g(t+c)\}$ | $e^{at}\sin(bt)$ | $\frac{b}{(s-a)^2+b^2}$ |
| $tf(t)$ | $-\frac{dF(s)}{ds}$ | $e^{at}\cos(bt)$ | $\frac{s-a}{(s-a)^2+b^2}$ |
| $t^n f(t)$, $n = 1, 2, 3, \ldots$ | $(-1)^n F^{(n)}(s) = \frac{(-1)^n d^n F(s)}{ds^n}$ | $e^{at}f(t)$ | $F(s-a)$ |
| $\frac{1}{t}f(t)$ | $\int_s^\infty F(u)\,du$ | $t^n e^{at}$, $n = 1,2,3,\ldots$ | $\frac{n!}{(s-a)^{n+1}}$ |

| | | | |
|---|---|---|---|
| $\int_0^t f(v)dv$ | $\frac{F(s)}{s}$ | $\sinh(at)$ | $\frac{a}{s^2-a^2}$ |
| $\int_0^t f(t-\tau)g(\tau)d\tau$ | $F(s)G(s)$ | $\cosh(at)$ | $\frac{s}{s^2-a^2}$ |
| $f(t+T)=f(t)$ | $\frac{\int_0^T e^{-st}f(t)dt}{1-e^{-sT}}$ | $e^{at}\sinh(bt)$ | $\frac{b}{(s-a)^2-b^2}$ |
| $f'(t)$ | $sF(s)-f(0)$ | $e^{at}\cosh(bt)$ | $\frac{s}{(s-a)^2-b^2}$ |
| $f''(t)$ | $s^2F(s)-sf(0)$ $-f'(0)$ | $\frac{\sin(at)}{t}$ | $\arctan\left(\frac{a}{s}\right)$ |
| $f^{(n)}(t)$ | $s^nF(s)-s^{n-1}f(0)-s^{n-2}f'(0)\ldots-sf^{(n-2)}(0)-f^{(n-1)}(0)$ | | |
| $\frac{1}{\sqrt{\pi t}}e^{-a^2/4t}$ | $\frac{e^{-a\sqrt{s}}}{\sqrt{s}}$ | $t\sinh(bt)$ | $\frac{2bs}{(s^2-b^2)^2}$ |
| $\frac{a}{2\sqrt{\pi t^3}}e^{-a^2/4t}$ | $e^{-a\sqrt{s}}$ | $t\cosh(bt)$ | $\frac{s^2+b^2}{(s^2-b^2)^2}$ |
| $erf\left(\frac{a}{2\sqrt{t}}\right)$ | $\frac{e^{a\sqrt{s}}}{s}$ | $\frac{e^{at}-e^{bt}}{a-b}$ | $\frac{1}{(s-a)(s-b)}$ |
| $te^{at}$ | $1/(s-a)^2$ | $\frac{ae^{at}-be^{bt}}{a-b}$ | $\frac{s}{(s-a)(s-b)}$ |
| $\frac{e^{-at}}{(b-a)(c-a)}+\frac{e^{-bt}}{(c-b)(a-b)}+\frac{e^{-ct}}{(a-c)(b-c)}$ | | | $\frac{1}{(s+a)(s+b)(s+c)}$ |
| $\frac{1}{b^2}(1-\cos(bt))$ | $\frac{1}{s(s^2+b^2)}$ | $\frac{1}{a^2}(at-1+e^{-at})$ | $\frac{1}{(s^2)(s+a)}$ |
| | | $\frac{1}{a^2}(1-e^{-at}-ate^{-at})$ | $\frac{1}{s(s+a)^2}$ |

$$\mathfrak{L}\{f(t)\}=\int_0^\infty e^{-st}f(t)dt$$

$$\Gamma(t)=\int_0^\infty e^{-x}x^{t-1}\,dx,\qquad \Gamma(n+1)=n!,\quad \Gamma\left(\frac{1}{2}\right)=\sqrt{\pi}$$

$$\operatorname{erf}(t)=\frac{2}{\sqrt{\pi}}\int_0^t e^{-x^2}dx$$