MTH 277, Final Exam, Spring 2018 Name KEY

Instructions: Show all work. Give exact answers unless specifically asked to round. All complex numbers should be stated in standard form, and all complex fractions should be simplified. If you do not show work, problems will be graded as "all or nothing" for the answer only; partial credit will not be possible and any credit awarded for the work will not be available.

1. Find $\frac{dw}{dt}$ for $w = xy^2 + x^2z + yz^2, x = t^2, y = 2t, z = \ln t$ using the chain rule. Be sure your final answer contains only t. (12 points)

$$\frac{\partial w}{\partial x} = y^2 + 2xz = (2t)^2 + 2(t^2)(\ln t) \qquad \frac{dx}{dt} = 2t$$

$$\frac{\partial w}{\partial y} = 2xy + z^2 = 2(t^2)(2t) + \ln^2 t \qquad \frac{dy}{dt} = 2$$

$$\frac{\partial w}{\partial z} = x^2 + 2yz = (t^2)^2 + 2(2t)\ln t \qquad \frac{dz}{dt} = \frac{1}{t}$$

$$\frac{dw}{dt} = [4t^2 + 2t^2\ln t](2t) + [4t^3 + \ln^2 t](2) + [t^4 + 4t\ln t]\cdot\frac{1}{t}$$

2. Find the three first implicit partial derivatives for $x^2y + y^2 + z^2\cos w - 5yw + 10w^2 = 2$. (15 points)

$$F = x^2y + y^2 + z^2\cos w - 5yw + 10w^2 - 2 = 0$$

$$F_x = 2xy$$

$$F_y = x^2 + 2y - 5w$$

$$F_z = 2z\cos w$$

$$F_w = -z^2\sin w - 5y + 20w$$

$$\frac{\partial w}{\partial x} = -\frac{2xy}{-z^2\sin w - 5y + 20w} \qquad \frac{\partial w}{\partial z} = \frac{-2z\cos w}{-z^2\sin w - 5y + 20w}$$

$$\frac{\partial w}{\partial y} = -\frac{x^2 + 2y - 5w}{-z^2\sin w - 5y + 20w}$$

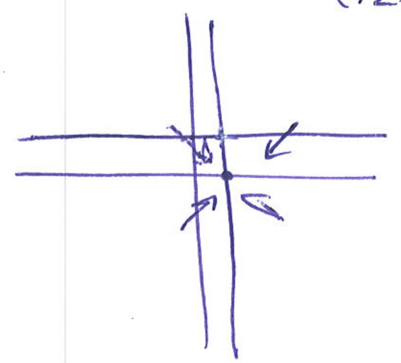
3. Sketch the gradient field for $z = -3x^2 - 2y^2 + 3x - 4y + 5$ and use it to characterize any critical points. (18 points)

$f_x = -6x + 3 = 0 \quad 6x = 3$
$x = 1/2$

$f_y = -4y - 4 = 0 \quad y = -1$

$(1/2, -1)$

(x, y)	∇f
$(0,0)$	$\langle 3, -4 \rangle$
$(0,-2)$	$\langle 3, 4 \rangle$
$(1, 0)$	$\langle -3, -4 \rangle$
$(1,-2)$	$\langle -3, 4 \rangle$



maximum

4. Use the second partials test to find any characterize any critical points for $z = -5x^2 + 4xy - y^2 + 16x + 10$. (12 points)

$z_x = -10x + 4y + 16 = 0$

$z_y = 4x - 2y = 0$

$4x = 2y$

$2x = y$

$-10x + 4(2x) = -16$

$-2x = -16$

$x = 8$

$y = 16$

$f_{xx} = -10$

$f_{yy} = -2$

$f_{xy} = 4$

$D = (-10)(-2) - 16 =$

$20 - 16 = 4 > 0$

$f_{xx} < 0 \quad \cap$

$(8, 16)$ is a max

5. Find the absolute extrema of the function $f(x,y) = 3x^2 - 2xy + y^2 - 4y$ over the region bounded by $y = 4 - x^2, and\ y = 0$. (18 points)

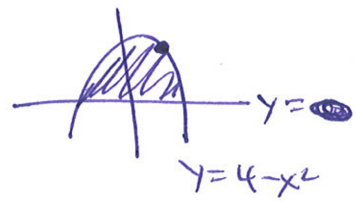
$f_x = 6x - 2y = 0 \qquad 6x = 2y \to y = 3x$

$f_y = -2x + 2y - 4 = 0$

$-2x + 2(3x) = 4$

$4x = 4$

$x = 1 \quad y = 3$



f

ABSMN $(1,3) \to -6$

ABSMAX $(0,-2) \to 12$

$(0,2) \to -4$

$(0,0) \to 0$

$f(x,0) = 3x^2 \to f'(x) = 6x = 0 \quad x = 0$

$f(x, 4-x^2) = 3x^2 - 2x(4-x^2) + (4-x^2)^2 - 4(4-x^2)$

$3x^2 - 8x + 2x^3 + \cancel{16} - 8x^2 + x^4 - \cancel{16} + 4x^2$

$x^4 + 2x^3 - x^2 - 8x$

$f'(x) = 4x^3 + 6x^2 - 2x - 8 = 2(2x^3 + 3x^2 - x - 4) = 0$

when $x = 1$ (from graph)

gives $(1,3)$ again

6. Find the volume of the region below $f(x,y) = \frac{xy}{1+x^2y^2}$ for the region bounded by $xy = 1, xy = 2, x = 1, x = 4$. (18 points)

$u = xy$

$v = x$

$u = vy$

$y = \frac{u}{v}$

$J = \begin{vmatrix} 0 & 1 \\ \frac{1}{v} & -\frac{u}{v^2} \end{vmatrix} = -\frac{1}{v} \qquad |J| = \frac{1}{v} \qquad \begin{matrix} 1 \le u \le 2 \\ 1 \le v \le 4 \end{matrix}$

$\int_1^4 \int_1^2 \frac{u}{1+u^2} \cdot \frac{1}{v} \, du \, dv =$

$\int_1^4 \frac{1}{v} dv \cdot \int_1^2 \frac{u}{1+u^2} du =$

$\ln v \Big|_1^4 \cdot \frac{1}{2} \ln |1+u^2| \Big|_1^2 =$

$(\ln 4 - \cancel{\ln 1}) \frac{1}{2} (\ln 5 - \ln 2) =$

$\boxed{\frac{1}{2} \ln 4 \left(\ln\left(\frac{5}{2}\right)\right)}$

0.63

7. Find the position vector for $\vec{a}(t) = e^t\hat{\imath} - 8\hat{k}, \vec{v}(0) = 2\hat{\imath} + 3\hat{\jmath} + \hat{k}, \vec{r}(0) = \vec{0}$. (15 points)

$v(t) = \int a(t)\,dt = (e^t + C_1)\hat{\imath} + C_2\hat{\jmath} + (-8t + C_3)\hat{k}$

$= 2$

$e^0 + C_1 = 2$ $\quad C_2 = 3 \quad -8(0) + C_3 = 1$

$C_1 = 1 \quad\quad C_3 = 1$

$v(t) = (e^t + 1)\hat{\imath} + 3\hat{\jmath} + (-8t + 1)\hat{k}$

$r(t) = \int v(t)\,dt = (e^t + t + C_1)\hat{\imath} + (3t + C_2)\hat{\jmath} + (-4t^2 + t + C_3)\hat{k}$

$e^0 + 0 + C_1 = 0 \quad 3(0) + C_2 = 0 \quad -4(0)^2 + 0 + C_3 = 0$

$C_1 = -1 \quad C_2 = 0 \quad 0 = C_3$

$r(t) = (e^t + t - 1)\hat{\imath} + (3t)\hat{\jmath} + (-4t^2 + t)\hat{k}$

8. Determine the maximum height and range of a projectile fired at height of 1.5 meters above the ground, with an initial velocity of 100 meters/second at an angle of 30° with the horizontal. Use the equation $\vec{r}(t) = (v_0 \cos\theta)t\hat{\imath} + \left[h_0 + (v_0 \sin\theta)t + \frac{1}{2}gt^2\right]\hat{\jmath}$. (15 points)

$(100\cos 30°)t\hat{\imath} + \left[1.5 + (100\sin 30°)t - \frac{9.8}{2}t^2\right]\hat{\jmath}$

$1.5 + (100\sin 30°)t - 4.9t^2 = 0$

$= 0$ when $t = 10.234$ sec

range $= (100\cos 30°)\,10.234 = 886.3$ meters

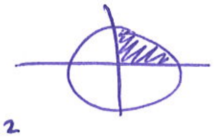
max height

$y'(t) = 100\sin 30° - 9.8t = 0$

$t = 5.102 \quad$ height max $= 129.05$

9. Write the integrals needed to find the center of mass for the region bounded by $0 \leq z \leq 12e^{-(x^2+y^2)}, x^2+y^2 \leq 4, x \geq 0, y \geq 0, \rho = k\sqrt{x^2+y^2}$. You do not need to integrate. (20 points)

$12e^{-r^2}$ $\quad r \leq 2 \quad$ $\rho = kr$



$$M = \int_0^{\pi/2} \int_0^2 \int_0^{12e^{-r^2}} kr \cdot r\,dz\,dr\,d\theta \qquad r^2$$

$$M_{xy} = \int_0^{\pi/2} \int_0^2 \int_0^{12e^{-r^2}} k \cdot r \cdot z \cdot r\,dz\,dr\,d\theta \qquad r^2 z$$

$$M_{yz} = \int_0^{\pi/2} \int_0^2 \int_0^{12e^{-r^2}} kr \cdot r\cos\theta \cdot r\,dz\,dr\,d\theta$$

$$r^3 \cos\theta$$

$$M_{xz} = \int_0^{\pi/2} \int_0^2 \int_0^{12e^{-r^2}} kr \cdot r\sin\theta \cdot r\,dz\,dr\,d\theta$$

$$r^3 \sin\theta$$

10. Find the curvature of the curve $\vec{r}(t) = 5t\hat{\imath} + \ln t\,\hat{\jmath} + 2t^2\hat{k}$ at the point $t = 1$. What is the radius of curvature at the same point? (14 points)

$$r'(t) = 5\hat{\imath} + \frac{1}{t}\hat{\jmath} + 4t\hat{k}$$

$$r''(t) = 0\hat{\imath} - \frac{1}{t^2}\hat{\jmath} + 4\hat{k}$$

$$r' \times r'' = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ 5 & \frac{1}{t} & 4t \\ 0 & -\frac{1}{t^2} & 4 \end{vmatrix} = \left(\frac{4}{t} + \frac{4t}{t^2}\right)\hat{\imath} - (20 - 0)\hat{\jmath} + \left(-\frac{5}{t^2} - 0\right)\hat{k}$$

$$\frac{8}{t}\hat{\imath} - 20\hat{\jmath} - \frac{5}{t^2}\hat{k}$$

$$\|r' \times r''\| = \sqrt{\frac{64}{t^2} + 400 + \frac{25}{t^4}} \qquad \text{at } t=1 \quad \sqrt{64+400+25} = \sqrt{489}$$

$$\|r'\| = \sqrt{25 + \frac{1}{t^2} + 16t^2} \qquad \text{at } t=1 \quad \sqrt{25+1+16} = \sqrt{42}$$

$$K = \frac{\sqrt{489}}{(42)^{3/2}} \approx .08 \qquad R = \frac{42^{3/2}}{\sqrt{489}} \approx 12.31$$

11. Find the equation of the tangent plane to the surface $x^2 + 2z^2 = y^2$ at $(1,3,-2)$. (12 points)

$X^2 + 2Z^2 - Y^2 = 0$

$\nabla f = \langle 2x, -2y, 4z \rangle$

$\langle 2, -6, -8 \rangle$

$2(x-1) - 6(y-3) - 8(z+2) = 0$

12. Use Green's Theorem to evaluate $\int_C (y - e^x)dx + (2x - \ln y)dy$ on the path described by the boundary of the graphs $y = x, y = x^2 - 2x$ oriented counterclockwise. (15 points)

$M = y - e^x \qquad N = 2x - \ln y$

$\frac{\partial M}{\partial y} = 1 \qquad \frac{\partial N}{\partial x} = 2$

$\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} = 2 - 1 = 1$

$\int_0^3 \int_{x^2-2x}^{x} 1 \, dy \, dx$

$\int_0^3 x - (x^2 - 2x) \, dx =$

$\int_0^3 3x - x^2 \, dx =$

$\frac{3}{2}x^2 - \frac{1}{3}x^3 \Big|_0^3 = \frac{27}{2} - 9 = \boxed{\frac{9}{2}}$

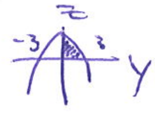
$X = X^2 - 2X$

$0 = X^2 - 3X$

$X = 0, X = 3$

13. Use the divergence theorem to evaluate $\int_S \int \vec{F} \cdot \vec{N} dS$ for $\vec{F}(x, y, z) = (4xy + z^2)\hat{\imath} + (2x^2 + 6yz)\hat{\jmath} + 2xz\hat{k}$ for the closed surface bounded by $x = 4, z = 9 - y^2$ and the coordinate planes. (20 points)

y



$$\nabla \cdot F = 4y + 6z + 2x$$

$$\int_0^4 \int_0^3 \int_0^{9-y^2} 4y + 6z + 2x \; dz\, dy\, dx =$$

$$\int_0^4 \int_0^3 4yz + 3z^2 + 2xz \Big|_0^{9-y^2} dy\, dx =$$

$$\int_0^4 \int_0^3 -2xy^2 + 3y^4 - 4y^3 - 27y^2 + 18x - 27y^2 + 36y + 243 \cdot dy\, dx$$

$$\int_0^4 36x + \frac{2349}{5} \; dx = \frac{10,836}{5} = 2167.2$$

14. Write an equation of the cylinder $4x^2 + z^2 = 16$ in parametric (surface) form. (10 points)

$$\frac{4x^2}{16} + \frac{z^2}{16} = \frac{16}{16}$$

$$\frac{x^2}{4} + \frac{z^2}{16} = 1$$

$a = 2 \quad b = 4 \qquad$ y is free

$$r(u, v) = 2\cos u \,\hat{\imath} + v\,\hat{\jmath} + 4\sin u\, \hat{k}$$

15. Find the limit $\lim_{(x,y)\to(0,0)} \frac{x^2y}{x^4+y^2}$. (10 points)

$y = kx^2$

$$\lim_{x\to 0} \frac{x^2 \cdot kx^2}{x^4 + k^2x^4} = \lim_{x\to 0} \frac{\cancel{x^4}\, k}{\cancel{x^4}(1+k^2)} = \frac{k}{1+k^2}$$

DNE
Since it depends on K

16. Convert the triple integral $\int_{-4}^{4}\int_{0}^{\sqrt{16-x^2}}\int_{0}^{\sqrt{16-x^2-y^2}} \sqrt{x^2+y^2+z^2}\, dzdydx$ to spherical coordinates and then complete the integration. Describe the region being integrated (over). (15 points)

Sphere radius 4
top half (hemisphere)
top half (circle in plane)

$$\int_0^{\pi}\int_0^{\pi}\int_0^{4} \rho\cdot\rho^2 \sin\varphi \, d\rho\, d\varphi\, d\theta$$

$$\rho^3 \sin\varphi \, d\rho\, d\varphi\, d\theta$$

$$\int_0^{\pi}\int_0^{\pi} \sin\varphi \tfrac{1}{4}\rho^4 \Big|_0^4 \, d\varphi\, d\theta = 64\int_0^{\pi}\int_0^{\pi} \sin\varphi \, d\varphi\, d\theta$$

$$64\int_0^{\pi} -\cos\varphi \Big|_0^{\pi} d\theta = 64\int_0^{\pi} -(-1-1)\, d\theta =$$

$$128\int_0^{\pi} d\theta = \boxed{128\pi}$$

17. Change the order of integration in $\int_0^4 \int_{y/4}^1 \frac{1}{1+x^4}\, dxdy$ so that it can be integrated. Then complete the integration. (12 points)

$x = y/4$

$y = 4x$

$$\int_0^1 \int_0^{4x} \frac{1}{1+x^4}\, dy\, dx = \int_0^1 \frac{y}{1+x^4}\Big|_0^{4x} dx =$$

$$\int_0^1 \frac{4x}{1+x^4}\, dx = 2\int_0^1 \frac{2x}{1+x^4}\, dx = 2 \arctan x^2 \Big|_0^1$$

$u = x^2$

$du = 2x\, dx$

$$2\arctan(1) - \cancel{2\arctan 0}^{\,0}$$

$$2\left(\frac{\pi}{4}\right) = \boxed{\frac{\pi}{2}}$$

18. Determine if the vector field $\vec{F}(x,y,z) = y^2 z\hat{\imath} + 2xyz\hat{\jmath} + xy^2\hat{k}$ is conservative. Find the potential function. (12 points)

$$\nabla \times F = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ y^2 z & 2xyz & xy^2 \end{vmatrix} = (\cancel{2xy - 2xy})^{0}\hat{\imath} - (\cancel{y^2 - y^2})^{0}\hat{\jmath}$$

$$+ (\cancel{2yz - 2yz})^{0}\hat{k}$$

$$= \vec{0}$$

Conservative

$$\int y^2 z\, dx = xy^2 z + f(y,z)$$

$$\int 2xyz\, dy = xy^2 z + g(x,z)$$

$$\int xy^2\, dz = xy^2 z + h(x,y)$$

$$\varphi = xy^2 z + K$$

19. Evaluate the line integral $\int_C 2xyz\,ds$ on the path $\vec{r}(t) = t\hat{\imath} + 5t\hat{\jmath} + 3t\hat{k}$ on $[0,1]$. (12 points)

$r'(t) = 1\hat{\imath} + 5\hat{\jmath} + 3\hat{k}$

$\|r'(t)\| = \sqrt{1+25+9} = \sqrt{35}$

$\int_0^1 2(t)(5t)(3t)\sqrt{35}\,dt$

$30\sqrt{35}\int_0^1 t^3\,dt = \frac{30\sqrt{35}}{4} t^4 \Big|_0^1 = \boxed{\frac{15\sqrt{35}}{2}}$