277 Homework #10 Key

1a. $w = x\sin y \qquad x = e^t \quad y = \pi - t$

$\frac{\partial w}{\partial x} = \sin y \qquad \frac{dx}{dt} = e^t$

$\frac{\partial w}{\partial y} = -x\cos y \qquad \frac{dy}{dt} = -1$

$$\frac{dw}{dt} = e^t \sin(\pi - t) + e^t \cos(\pi - t)$$
$$= e^t\left(\sin(\pi - t) + \cos(\pi - t)\right)$$

$$\frac{d^2w}{dt^2} = e^t\left(\sin(\pi - t) + \cos(\pi - t)\right) +$$
$$e^t\left(-\cos(\pi - t) + \sin(\pi - t)\right)$$

$$= 2e^t \sin(\pi - t)$$

b. $w = \cos(x - y)$

$x = t^2, \; y = 1$

$\frac{\partial w}{\partial x} = -\sin(x-y) = -\sin(t^2 - 1) \qquad \frac{dx}{dt} = 2t \qquad \frac{dw}{dt} = -2t\sin(t^2-1) + 0$

$\frac{\partial w}{\partial y} = \sin(x - y) = \sin(t^2 - 1) \qquad \frac{dy}{dt} = 0$

$$\frac{d^2w}{dt^2} = -2\sin(t^2-1) -$$
$$2t\cos(t^2-1)(2t)$$
$$= 2\sin(t^2-1) - 4t^2\cos(t^2-1)$$

2a. $w = x^2 + y^2, \; x = s + t, \; y = s - t$

$\frac{\partial w}{\partial x} = 2x = 2(s+t) \qquad \frac{\partial x}{\partial t} = 1 \qquad \frac{\partial x}{\partial s} = 1$

$\frac{\partial w}{\partial y} = 2y = 2(s-t) \qquad \frac{\partial y}{\partial t} = -1 \qquad \frac{\partial y}{\partial s} = 1$

$$\frac{\partial w}{\partial t} = 2(s+t)(1) + 2(s-t)(-1)$$
$$\frac{\partial w}{\partial s} = 2(s+t)(1) + 2(s-t)(1)$$

b. $w = xyz, \quad x = s + t, \; y = s - t, \; z = st^2$

$\frac{\partial w}{\partial x} = yz = (s-t)(st^2) \qquad \frac{\partial x}{\partial t} = 1 \qquad \frac{\partial x}{\partial s} = 1$

$\frac{\partial w}{\partial y} = xz = (s+t)(st^2) \qquad \frac{\partial y}{\partial t} = -1 \qquad \frac{\partial y}{\partial s} = 1$

$\frac{\partial w}{\partial z} = xy = (s+t)(s-t) \qquad \frac{\partial z}{\partial t} = 2st \qquad \frac{\partial z}{\partial s} = t^2$

$$\frac{\partial w}{\partial t} = (s-t)(st^2)(1) + (s+t)(st^2)(-1) + (s+t)(s-t)(2st)$$

$$\frac{\partial w}{\partial s} = (s-t)(st^2)(1) + (s+t)(st^2)(1) + (s+t)(s-t)t^2$$

2c. $w = y^3 - 3x^2y$, $x = e^s$, $y = e^{t^2}$

$\frac{\partial x}{\partial t} = 0$ $\quad \frac{\partial y}{\partial t} = 2te^{t^2}$

$\frac{\partial x}{\partial s} = e^s$ $\quad \frac{\partial y}{\partial s} = 0$

$$\frac{\partial w}{\partial x} = -6xy = -6e^s \cdot e^{t^2} = -6e^{s+t^2}$$

$$\frac{\partial w}{\partial y} = 3y^2 - 3x^2 = 3(e^{t^2})^2 - 3(e^s)^2 = 3e^{2t^2} - 3e^{2s}$$

$$\frac{\partial w}{\partial t} = (-6e^{s+t^2})(0) + [3e^{2t^2} - 3e^{2s}]e^{t^2} \cdot 2t$$

$$\frac{\partial w}{\partial s} = (-6e^{s+t^2})e^s + [3e^{2t^2} - 3e^{2s}](0)$$

3\. a. $\frac{\partial z}{\partial x} = \frac{-(6x - y + z + \frac{z}{x^2})}{x - 6yz^2 + 4z^3 - \frac{1}{x}}$ $\quad F = 3x^2 - xy + xz - 2yz^3 + z^4 - \frac{z}{x} = 0$

$$\frac{\partial z}{\partial y} = \frac{-(-x - 2z^3)}{x - 6yz^2 + 4z^3 - \frac{1}{x}}$$

b. $F = x^5 - xyz + z^4 - \ln x + \ln y = 0$

$$\frac{\partial z}{\partial x} = \frac{-(5x^4 - yz - \frac{1}{x})}{-xy + 4z^3}$$

$$\frac{\partial z}{\partial y} = \frac{-(-xz + \frac{1}{y})}{-xy + 4z^3}$$

c. $F = 4x^3 - \sin(xy) - ye^x$

$$\frac{\partial y}{\partial x} = \frac{-(12x^2 - y\cos(xy) - ye^x)}{-x\cos(xy) - e^x}$$

3d. $F = 2x^2w - xyz - 2w\tan(zw) + e^{4w} - \frac{zy}{w} = 0$

$$\frac{\partial w}{\partial x} = \frac{-(4xw - yz)}{2x^2 - 2\tan(zw) - 2zw\sec^2(zw) + 4e^{4w} + \frac{zy}{w^2}}$$

$$\frac{\partial w}{\partial y} = \frac{-(-xz - \frac{z}{w})}{2x^2 - 2\tan(zw) - 2zw\sec^2(zw) + 4e^{4w} + \frac{zy}{w^2}}$$

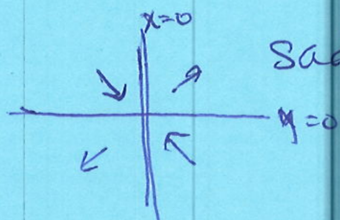
$$\frac{\partial w}{\partial z} = \frac{-(xy - 2w^2\sec^2(zw) - \frac{y}{w})}{2x^2 - 2\tan(zw) - 2zw\sec^2(zw) + 4e^{4w} + \frac{zy}{w^2}}$$

4. See attached

5. See attached

6. a. $f_x = y = 0$

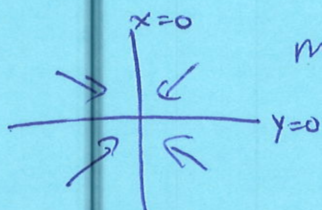
$f_y = x = 0$



saddle at (0,0).

b. $k_x = -2x = 0$ $x = 0$

$k_y = -8y = 0$ $y = 0$



max at (0,0)

c. $p_x = \frac{1(x^2+y) - 2x(x+y)}{(x^2+y)^2} = \frac{x^2+y-2x^2-2xy}{(x^2+y)^2} = \frac{-x^2+y-2xy}{(x^2+y)^2}$

$y(1-2x) = x^2$

$y = \frac{x^2}{1-2x}$

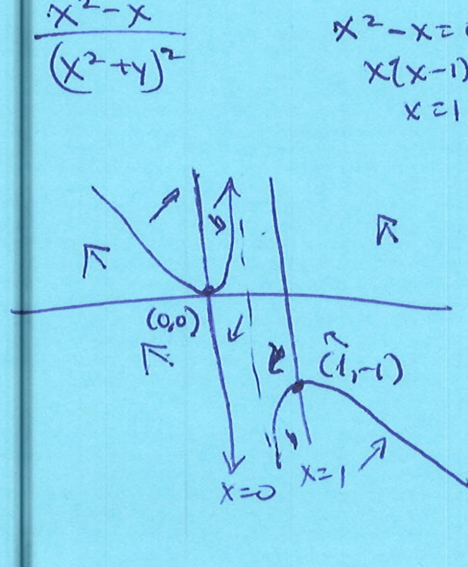
$p_y = \frac{1(x^2+y) - 1(x+y)}{(x^2+y)^2} = \frac{x^2+\cancel{y}-x-\cancel{y}}{(x^2+y)^2} = \frac{x^2-x}{(x^2+y)^2}$

$x^2 - x = 0$

$x(x-1) = 0$

$x = 1, 0$

critical points at (0,0)
and (1,-1)



MTH 277, Homework #10, Spring 2018 Name ______________________________

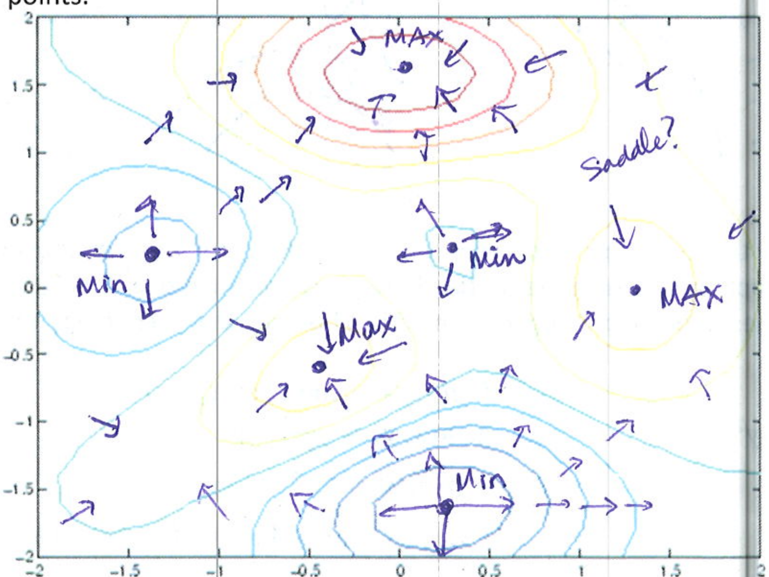
Instructions: Write your work up neatly and attach to this page. Record your final answers (only) directly on this page if they are short; if too long indicate which page of the work the answer is on and mark it clearly. Use exact values unless specifically asked to round.

1. Find $\frac{dw}{dt}$ for the following sets of equations using the chain rule. Be sure your final answers contain onl C nd $\frac{d^2w}{dt^2}$ for both.
   a. $w = xs$ $x = e^t, y = \pi - t$ b. $w = \cos(x - y), x = t^2, y = 1$

2. Find $\frac{\partial w}{\partial t}$ and $\frac{\partial w}{\partial s}$ for the following sets of equations using the chain rule. Be sure your final answer contains only t and s.
   a. $w = x^2 + y^2, x = s + t, y = s - t$ c. $w = y^3 - 3x^2y, x = e^s, y = e^{t^2}$
   b. $w = xyz, x = s + t, y = s - t, z = st^2$

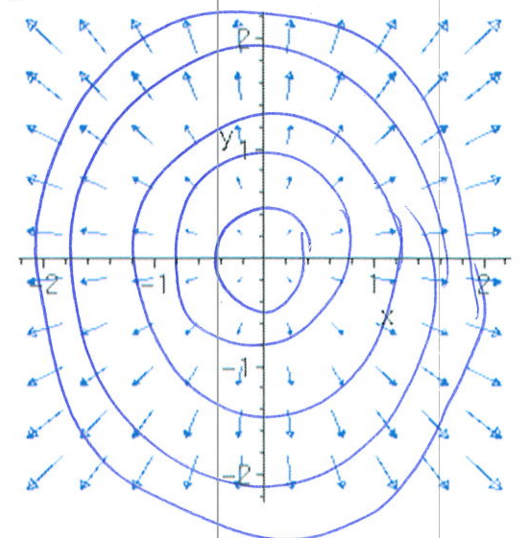
3. Find the implicit derivative, or first partial derivatives of the following implicit functions. Calculate the derivatives the "long way" and using the formula. Verify that both produce the same result. Don't find more than you need. Simplify any complex fractions.
   a. $3x^2 - xy + xz - 2yz^3 + z^4 = \frac{z}{x}$ c. $4x^3 - \sin(xy) = ye^x$
   b. $x^5 - xyz + z^4 = \ln\left(\frac{x}{y}\right)$ d. $2x^2w - xyz - 2w\tan(zw) + e^{4w} = \frac{zy}{w}$

4. Use the idea that gradients are perpendicular to level curves to sketch the gradient field for the graph below. Assume that blues are lower values, and reds are larger values to determine the direction of your vectors. Note any critical points or other interesting features, including saddle points.

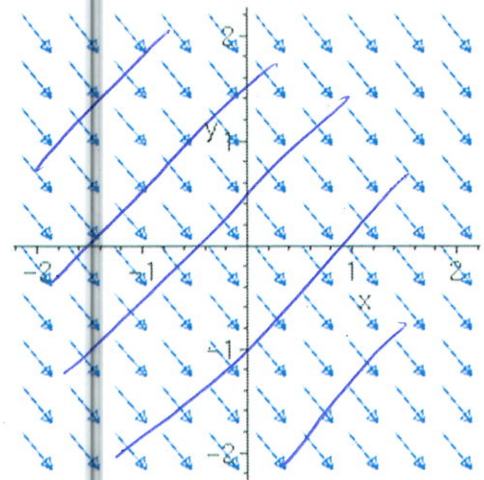


5. For each of the gradient fields below, graph at least 5 level curves using the fact that the gradient is perpendicular to the level curves. Use that information to determine if the critical points are maxima, minima, or saddle points.

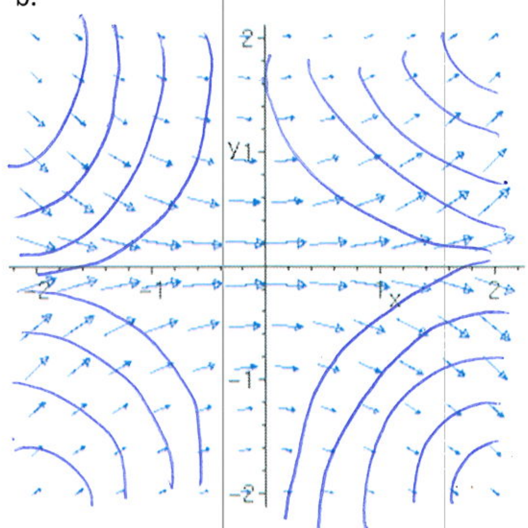
a.



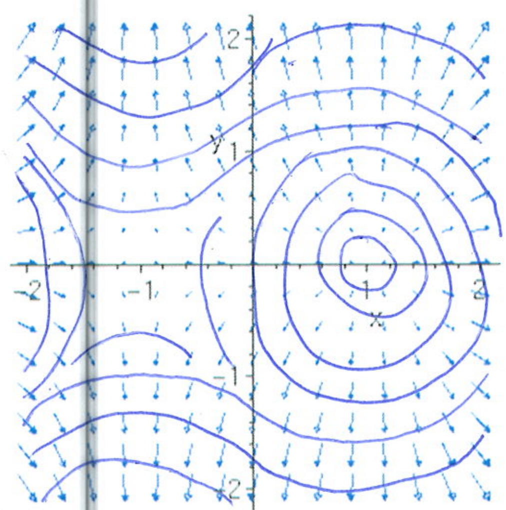
c.



b.



d.



6. Following the method in the Gradients and Level Curves handout, for each of the following equations, analyze the graph by graphing the partial derivatives and analyzing the gradient field in each region of the plane. Label and categorize any critical points. Use technology to confirm your results.

a. $f(x,y) = xy$

b. $k(x,y) = 4 - x^2 - 4y^2$

c. $p(x,y) = \frac{x+y}{x^2+y}$

d. $r(x,y) = \frac{1}{x} + \frac{1}{y} + xy$

e. $t(x,y) = 4x^2 - y$

f. $g(x,y) = \sin(x)\cos(y)$

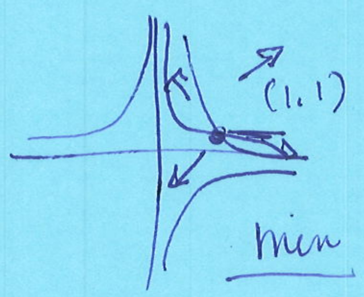
g. $m(x,y) = \frac{x}{1+x^2+y^2}$

h. $q(x,y) = 3x^3 - 12xy + y^3$

i. $s(x,y) = x^2 + 4xy + y^2 - 4x + 16y + 3$

6d. $r_x = -\frac{1}{x^2} + y = 0 \qquad y = \frac{1}{x^2}$

$r_y = -\frac{1}{y^2} + x = 0 \qquad x = +\frac{1}{y^2},\ y^2 = \frac{1}{x} \to y = \frac{1}{\pm\sqrt{x}}$

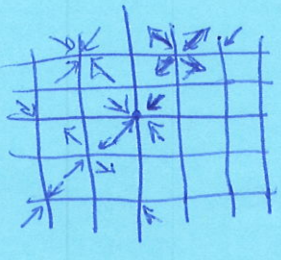


e. $t_x = 8x \qquad x = 0$

$t_y = -1$ never 0   critical pt(s) = line

f. $g_x = \cos x \cos y \qquad x = \frac{\pi}{2}, \frac{3\pi}{2},\ y = \frac{\pi}{2}, \frac{3\pi}{2}$

$g_y = -\sin x \sin y \qquad x = 0, \pi, 2\pi \quad y = 0, \pi, 2\pi$

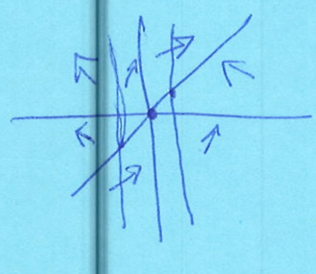


every other point is a min/saddle/max/saddle, etc.

g. $m_x = \frac{1(1+x^2+y^2) - x(2x)}{(1+x^2+y^2)^2}$ in any direction

$= \frac{-x^2+y^2+1}{(1+x^2+y^2)^2}$

$m_y = \frac{1(1+x^2+y^2) - x(2y)}{(1+x^2+y^2)^2}$



line of critical points $x = y$

$-x^2+y^2+1=0 \qquad x^2 = 1+y^2 \qquad 2x^2 = 1$

$\underbrace{1+x^2+y^2}-2xy=0 \qquad x = \pm\frac{1}{\sqrt{2}}$

$2x^2 - 2xy = 0 \qquad 2x(x-y) = 0$

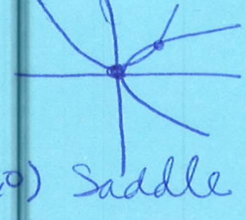
$x = 0,\ y = x \qquad x=0 \to \cancel{y = \pm i}$

h. $q_x = 9x^2 - 12y = 0 \qquad \frac{9x^2}{12} = y$

$q_y = -12x + 3y^2$

$3y^2 = \frac{12x}{3}$

$y = \pm\sqrt{4x}$

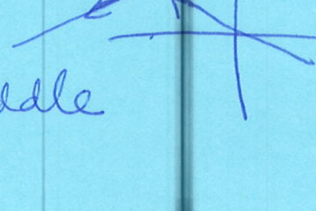


critical pt at (0,0) saddle

and ≈ (1.92, 2.77) min

i. $s_x = 2x + 4y - 4 = 0$

$s_y = 4x + 2y + 16 = 0$



(-6,4) saddle

7a. $f_x = 2x + y = 0 \quad y = -2x$

$f_{xx} = 2$

$f_y = x + 2y + 1 = 0 \quad\quad x + 2(-2x) = -1$

$f_{yy} = 2 \quad\quad x - 4x = -1$

$-3x = -1$

$f_{xy} = 1 \quad\quad x = \frac{1}{3}$

$y = -\frac{2}{3}$

$D = 2(2) - 1^2 = 4 - 1 = 3 > 0$

$f_{xx} > 0 \quad \cap \quad$ min @ $\left(\frac{1}{3}, -\frac{2}{3}\right)$

b. $f_x = 6xy - 12x = 0 \quad\quad 6x(y-2) = 0$

$f_{xx} = 6y - 12 \quad\quad x = 0$

$y = 2$

$f_y = 3y^2 + 3x^2 - 12y = 0 \quad\quad x = 0 \quad 3y^2 - 12y = 0$

$f_{yy} = 6y - 12 \quad\quad 3y(y-4) = 0$

$f_{xy} = 6x \quad\quad y = 2 \quad 12 + 3x^2 - 24 = 0$

$3x^2 = 12$

$x^2 = 4$

$x = \pm 2$

(0,0) MAX

(0,4) MIN

(2,2) SADDLE

(-2,2) SADDLE

$D(0,0) = (-12)(-12) - 0^2 = 144 > 0$

$f_{xx} < 0 \quad \cup$

$D(0,4) = (12)(12) - 0 = 144 > 0$

$f_{xx} < 0$

$D(2,2) = (0)(0) - 12^2 = -144 < 0$

$D(-2,2) = (0)(0) - (-12)^2 = -144 < 0$

c. $f_x = y - 2xy - y^2 = 0 \quad\quad f = xy - x^2y - xy^2$

$f_{xx} = -2y \quad\quad y(1 - 2x - y) = 0$

$y = 0$

$f_y = x - x^2 - 2xy = 0 \quad\quad y = 1 - 2x$

$f_{yy} = -2x \quad\quad x(1 - x - 2y) = 0$

$f_{xy} = 1 - 2x - 2y \quad\quad x = 0$

(0,0) SADDLE

(1,0) SADDLE

(0,1) SADDLE

$\left(\frac{1}{3}, \frac{1}{3}\right)$ MAX

$D(0,0) = (0)(0) - 1^2 = -1 < 0 \quad\quad y = 0 \quad 1 - x = 0$

$x = 1$

$D(1,0) = (0)(-2) - (1)^2 = -1 < 0 \quad\quad 1 - x - 2(1 - 2x) = 0$

$D(0,1) = (-2)(0) - (1)^2 = -1 < 0 \quad\quad 1 - x - 2 + 4x = 0$

$3x = 1$

$D\left(\frac{1}{3}, \frac{1}{3}\right) = \left(-\frac{2}{3}\right)\left(-\frac{2}{3}\right) - \left(\frac{1}{3}\right)^2 = \frac{1}{3} > 0 \quad\quad x = \frac{1}{3}$

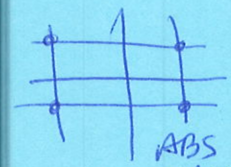
$f_{xx} < 0 \quad \cap \quad\quad y = 1 - 2\left(\frac{1}{3}\right)$

$= \frac{1}{3}$

8a. $f_x = 2x + y = 0$
$f_y = x = 0$

$f(2,y) = 4 + 2y$
$f'(y) = 2 \neq 0$
$f(-2,y) = 4 - 2y$
$f'(y) = -2 \neq 0$

$f(x,1) = x^2 + x$
$f'(x) = 2x + 1 = 0$
$x = -\frac{1}{2}$
$f(x,-1) = x^2 - x$
$f'(x) = 2x - 1$
$x = \frac{1}{2}$

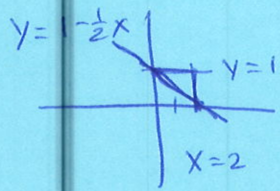


ABS MAX: $(2,1) \to 6$, $(2,-1) \to 2$, $(-2,1) \to 2$, $(-2,-1) \to 6$
$(0,0) \to 0$
ABS MIN: $(-\frac{1}{2},1) \to -\frac{1}{4}$, $(\frac{1}{2},-1) \to -\frac{1}{4}$

b. $f_x = -3 \neq 0$
$f_y = -2 \neq 0$

$f(x,1) = 12 - 3x - 2$
$f'(x) = -3 \neq 0$
$f(2,y) = 12 - 6 - 2y$
$f'(y) = -2 \neq 0$

$f(x, 1 - \frac{1}{2}x) =$
$12 - 3x - 2(1 - \frac{1}{2}x)$
$12 - 3x - 2 + x$
$10 - 2x$
$f'(x) = -2 \neq 0$

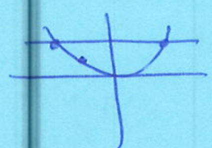


ABS MAX $f(0,1) = 12 - 0 - 2 = 10$
$f(2,0) = 12 - 6 - 0 = 6$
ABS MIN $f(1,2) = 12 - 3 - 4 = 5$

c. $f_x = 2 - 2y = 0 \quad y = 1$
$f_y = -2x + 2y \quad x = y \to x = 1 \ (1,1)$

$f(x,x^2) = 2x - 2x^3 + x^4$
$f'(x) = 2 - 6x^2 + 4x^3 = 0$
$x = 1 \to (1,1)$
$x = -\frac{1}{2} \quad (-\frac{1}{2}, \frac{1}{4})$

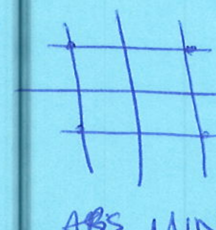
$f(x,1) = 2x - 2x + 1 = 1$
$f'(x) = 0$



$(-1,1) \to 1$ MAX
$(1,1) \to 1$
$(-\frac{1}{2}, \frac{1}{4}) \to -11/16$ MIN

d. $f_x = 2x + 2xy = 0 \quad 2x(1+y) = 0$
$x = 0, \ y = -1$
$f_y = 2y + x^2$
$2y = -x^2$
$y = 0$

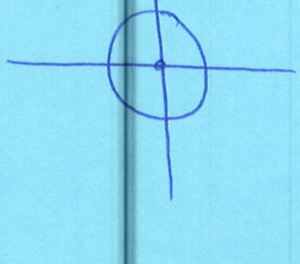
$+2 = +x^2 \to x = \pm\frac{1}{\sqrt{2}}$



$(1,1) \to 7$ ABS MAX
$(1,-1) \to 5$
$(-1,1) \to 7$ ABS MAX
$(-1,-1) \to 5$
ABS MIN $\Leftarrow (0,0) \to 4$
$(\frac{1}{\sqrt{2}},-1) \to 5$
$(-\frac{1}{\sqrt{2}},-1) \to 5$

e. $f_x = 6x^2 \Rightarrow x = 0$
$f_y = 4y^3 \Rightarrow y = 0$

$f(x, \sqrt{1-x^2}) = 2x^3 + (1-x^2)^2 =$
$2x^3 + 1 - 2x^2 + x^4$
$f'(x) = 6x^2 - 4x + 4x^3$
$x = 0 \quad 2x(3x - 2 + 2x^2)$
$x = -2$ not on circle
$x = 1/2$
$y = \pm\frac{\sqrt{3}}{2}$



$(0,0) \to 0 \Leftarrow$ ABS MIN
$(0,1) \to 1$ ABS MAX
$(0,-1) \to 1$ ABS MAX
~~(2,1)~~
$(1/2, \frac{\sqrt{3}}{2}) \to 13/16$
$(1/2, -\frac{\sqrt{3}}{2}) \to 13/16$

9a. saddle point

b. left = min, center = saddle, right = min

c. min

d. left = min, right = max

10. a. $c = x + 3y + 5z$ planes

b. $c = y^2 + z^2$ cylinders wrapped around x-axis w/ different radii

c. $c = x^2 + 3y^2 + 5z^2$ ellipsoids

d. $c = x^2 - y^2 - z^2$ hyperboloids of one-sheet or 2 sheets w/ diff $\pm$ values of c.

11. $x + y + z = 12$

$x^2 + y^2 + z^2 = f(x,y,z)$ min.

$z = 12 - x - y$

$f(x,y) = x^2 + y^2 + (12 - x - y)^2$

$f_x = 2x + (12 - x - y)(2)(-1) = 0$

$f_y = 2y + (12 - x - y)(2)(-1) = 0$

$2x - 2y = 0 \quad x = y$

$\cancel{2}x = \cancel{2}(12 - x - y)$

$x = 12 - x - y$

$2x = 12 - y$

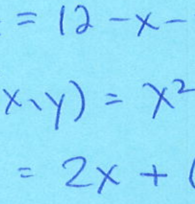
$2x = 12 - x$

$3x = 12$

$x = 4,\ y = 4$

(4, 4, 4)

12.



$z=0,\ y=0$

$y=0\quad 6 - 3z = 0 \qquad z = 2$

$z=0\quad 6 - 2y = 0 \qquad y = 3$

$V = xyz$ max

$x = 6 - 2y - 3z$

$(2, 1, 2/3)$

$V(y,z) = (6 - 2y - 3z)(yz) = 6yz - 2y^2z - 3yz^2$

$V_y = 6z - 4yz - 3z^2 = 0 \qquad z(6 - 4y - 3z) = 0$

$V_z = 6y - 2y^2 - 6yz = 0 \qquad y(6 - 2y - 6z) = 0$

$(6 - 4y = 3z)2$

$6 - 2y = 6z$

$12 - 8y = 6 - 2y$

$6 = 6y$

$y = 1$

$z = \frac{2}{3}$

$6 - 2(1) - 3(\frac{2}{3}) =$

$6 - 2 - 2 = 2$

$x = 2$

13. $d = \sqrt{(x-2)^2 + y^2 + (z-3)^2}$

$= \sqrt{(x-2)^2 + y^2 + (1-x-y-3)^2}$

$d_x = \frac{1}{2}\left[(x-2)^2 + y^2 + (1-x-y-3)^2\right]^{-1/2}\left(2(x-2) + 2(1-x-y-3)\right)$

$d_y = \frac{1}{2}\left[(x-2)^2 + y^2 + (1-x-y-3)^2\right]^{-1/2}\left(2y + 2(1-x-y-3)\right)$

$\cancel{2x} - 4 + 2 - \cancel{2x} - 2y - 6 = 0$

$-8 - 2y = 0$

$-8 = 2y$

$y = -4$

$\cancel{2y} + 2 - 2x - \cancel{2y} - 6 = 0$

$-4 = 2x$

$x = -2$

$z = 1 - (-2) - (-4) = 1 + 2 + 4 = 7$

$(-2, -4, 7)$

14. $SA = 2xy + 2xz + 2yz = 1500$

$V = xyz$

$4x + 4y + 4z = \text{Edges} = 200$

$g(x) = xy + xz + yz - 750$

$h(x) = x + y + z - 50$

λg_x etc $\quad \mu h_x$ etc

$f_x = yz = \lambda(y+z) + \mu$

$f_y = xz = \lambda(x+z) + \mu$

$f_z = xy = \lambda(x+y) + \mu$

$\mu = yz - \lambda(y+z)$

$\mu = xz - \lambda(x+z)$

$\mu = xy - \lambda(x+y)$

$yz - \lambda(y+z) = xz - \lambda(x+z)$

$yz - xz = \lambda(y+\cancel{z}) - \lambda(x+\cancel{z})$

$z(y-x) = \lambda(y-x) \quad y-x=0$

$z = \lambda$

$x = y$ or $z = \lambda$

$xz - \lambda(x+z) = xy - \lambda(x+y)$

$x(z-y) = \lambda(z-y) \quad z=y$ or $x=\lambda$

$x = y = z$

$x + y + z = 50$

$\Rightarrow 3z = 50$

$z = x = y = \frac{50}{3}$

$\left(\frac{50}{3}, \frac{50}{3}, \frac{50}{3}\right)$