MTH 277 Homework #7 key

1. a. $r'(t) = -6\sin t\,\hat{i} + 2\cos t\,\hat{j}$

$$\|r'(t)\| = \sqrt{36\sin^2 t + 4\cos^2 t} = \sqrt{32\sin^2 t + 4} = 2\sqrt{8\sin^2 t + 1}$$

$36 = (32+4)\sin^2 t$

$$T(t) = \frac{-6\sin t\,\hat{i} + 2\cos t\,\hat{j}}{\sqrt{32\sin^2 t + 4}} = \frac{-3\sin t\,\hat{i} + \cos t\,\hat{j}}{\sqrt{8\sin^2 t + 1}}$$

$$T'(t) = \left[(8\sin^2 t + 1)^{-1/2}(-3\sin t\,\hat{i} + \cos t\,\hat{j})\right]' =$$

$$-\frac{1}{\cancel{2}}(8\sin^2 t + 1)^{-3/2}\,\cancel{2}\cdot 8\sin t\cos t\,(-3\sin t\,\hat{i} + \cos t\,\hat{j}) + (8\sin^2 t + 1)^{-1/2}\cdot(-3\cos t\,\hat{i} - \sin t\,\hat{j})$$

$$= \frac{-8\sin t\cos t(-3\sin t\,\hat{i} + \cos t\,\hat{j})}{(8\sin^2 t + 1)^{3/2}} + \frac{-3\cos t\,\hat{i} - \sin t\,\hat{j}}{\sqrt{8\sin^2 t + 1}}\cdot\frac{8\sin^2 t + 1}{8\sin^2 t + 1}$$

$$= \frac{-8\sin t\cos t(-3\sin t\,\hat{i} + \cos t\,\hat{j}) - 24\sin^2 t\cos t\,\hat{i} - 3\cos t\,\hat{i} - 8\sin^3 t\,\hat{j} - \sin t\,\hat{j}}{(8\sin^2 t + 1)^{3/2}}$$

$-8\sin t(\cancel{\cos^2 t + \sin^2 t})$

$$= \frac{\cancel{24\sin^2 t\cos t\,\hat{i}} - \cancel{24\sin^2 t\cos t\,\hat{i}} - 3\cos t\,\hat{i} - 8\sin t\cos^2 t\,\hat{j} - 8\sin^3 t\,\hat{j} - \sin t\,\hat{j}}{(8\sin^2 t + 1)^{3/2}}$$

$$= \frac{-3\cos t\,\hat{i} - 9\sin t\,\hat{j}}{(8\sin^2 t + 1)^{3/2}}$$

$81 = 72 + 9(\cos^2 t)$

$$\|T'(t)\| = \frac{\sqrt{9\cos^2 t + 81\sin^2 t}}{(8\sin^2 t + 1)^{3/2}} = \frac{\sqrt{9 + 72\sin^2 t}}{(8\sin^2 t + 1)^{3/2}} = \frac{3\sqrt{1 + 8\sin^2 t}}{(8\sin^2 t + 1)^{3/2}}$$

$$N(t) = \frac{-\cancel{3}^{1}\cos t\,\hat{i} - \cancel{9}^{3}\sin t\,\hat{j}}{\cancel{(8\sin^2 t + 1)^{3/2}}}\cdot\frac{(8\sin^2 t + 1)^{3/2}}{\cancel{3}\sqrt{8\sin^2 t + 1}} = \frac{-\cos t\,\hat{i} - 3\sin t\,\hat{j}}{\sqrt{8\sin^2 t + 1}}$$

$$T(\pi/3) = \frac{-3\frac{\sqrt{3}}{2}\,\hat{i} + \frac{1}{2}\,\hat{j}}{\sqrt{7}} = \frac{-3\sqrt{3}\,\hat{i} + 1\,\hat{j}}{2\sqrt{7}}$$

$$N(\pi/3) = \frac{-\frac{1}{2}\hat{i} - \frac{3\sqrt{3}}{2}\hat{j}}{\sqrt{7}} = \frac{-\hat{i} - 3\sqrt{3}\,\hat{j}}{2\sqrt{7}}$$

②

16. $r'(t) = \frac{1}{t}\hat{\imath} + 1\hat{\jmath}$

$$\|r'(t)\| = \sqrt{\frac{1}{t^2}+1} = \sqrt{\frac{t^2+1}{t^2}} = \frac{1}{t}\sqrt{t^2+1}$$

$$T(t) = \frac{\frac{1}{t}\hat{\imath} + 1\hat{\jmath}}{\frac{1}{t}\sqrt{t^2+1}} \cdot \frac{t}{t} = \frac{1\hat{\imath} + t\hat{\jmath}}{\sqrt{t^2+1}} \qquad T(2) = \frac{\hat{\imath}+2\hat{\jmath}}{\sqrt{5}}$$

$$T'(t) = \left[(t^2+1)^{-1/2}(\hat{\imath}+t\hat{\jmath})\right]' = -\frac{1}{\cancel{2}}(t^2+1)^{-3/2}\cancel{2}t(\hat{\imath}+t\hat{\jmath}) + (t^2+1)^{-1/2}(\hat{\jmath})$$

$$= \frac{-t\hat{\imath} - t^2\hat{\jmath}}{(t^2+1)^{3/2}} + \frac{\hat{\jmath}}{(t^2+1)^{1/2}}\frac{(t^2+1)}{(t^2+1)} = \frac{-t\hat{\imath} - \cancel{t^2\hat{\jmath}} + \cancel{t^2\hat{\jmath}} + \hat{\jmath}}{(t^2+1)^{3/2}} =$$

$$\frac{-t\hat{\imath}+\hat{\jmath}}{(t^2+1)^{3/2}}$$

$$\|T'(t)\| = \frac{\sqrt{t^2+1}}{(t^2+1)^{3/2}} \qquad N(t) = \frac{-t\hat{\imath}+\hat{\jmath}}{\cancel{(t^2+1)^{3/2}}} \cdot \frac{\cancel{(t^2+1)^{3/2}}}{\sqrt{t^2+1}} = \frac{-t\hat{\imath}+\hat{\jmath}}{\sqrt{t^2+1}}$$

$$N(2) = \frac{-2\hat{\imath}+\hat{\jmath}}{\sqrt{5}}$$

c. $r'(t) = 4\hat{\imath} - 4\hat{\jmath} + 2\hat{k}$

$$\|r'(t)\| = \sqrt{16+16+4} = \sqrt{36} = 6$$

$$T(t) = \frac{4\hat{\imath} - 4\hat{\jmath} + 2\hat{k}}{6} = \frac{2}{3}\hat{\imath} - \frac{2}{3}\hat{\jmath} + \frac{1}{3}\hat{k} \quad \text{same for all } t$$

$N(t) = \vec{0}$ since $T'(t) = 0\hat{\imath} + 0\hat{\jmath} + 0\hat{k}$

This is a straight line ($r(t)$ is) and so any vector $\perp$ to the line is a normal vector. There is no principal normal vector.

d. $r'(t) = 2\cos t\,\hat{\imath} - 2\sin t\,\hat{\jmath} \qquad t = \pi/4$

$$\|r'(t)\| = \sqrt{4\cos^2 t + 4\sin^2 t} = \sqrt{4} = 2$$

$$T(t) = \frac{2\cos t\,\hat{\imath} - 2\sin t\,\hat{\jmath}}{2} = \cos t\,\hat{\imath} - \sin t\,\hat{\jmath} \qquad T(\pi/4) = \frac{1}{\sqrt{2}}\hat{\imath} - \frac{1}{\sqrt{2}}\hat{\jmath}$$

$$T'(t) = -\sin t\,\hat{\imath} - \cos t\,\hat{\jmath} \qquad \|T'(t)\| = 1 \qquad N(t) = -\sin t\,\hat{\imath} - \cos t\,\hat{\jmath}$$

$$N(\pi/4) = -\frac{1}{\sqrt{2}}\hat{\imath} - \frac{1}{\sqrt{2}}\hat{\jmath} \qquad \vec{B} = T \times N = -\hat{k}$$

1e. $r'(t) = (3t^2-4)\hat{\imath} + 4t\hat{\jmath}$

$$\|r'(t)\| = \sqrt{9t^4-24t^2+16+16t^2} = \sqrt{9t^4-8t^2+16}$$

$$T(t) = \frac{(3t^2-4)\hat{\imath} + 4t\hat{\jmath}}{\sqrt{9t^4-8t^2+16}} \qquad T(1) = \frac{-\hat{\imath}+4\hat{\jmath}}{\sqrt{17}}$$

$$T'(t) = \left[(9t^4-8t^2+16)^{-1/2} \left[(3t^2-4)\hat{\imath} + 4t\hat{\jmath}\right] \right]' =$$

$$\left(-\tfrac{1}{\cancel{2}}\right)(9t^4-8t^2+16)^{-3/2}(9\cdot \cancel{4}^{2}t^3 - \cancel{16}^{8}t)\left[(3t^2-4)\hat{\imath} + 4t\hat{\jmath}\right] +$$

$$(9t^4-8t^2+16)^{-1/2}\left[(6t)\hat{\imath} + 4\hat{\jmath}\right] =$$

$$\frac{(-18t^3+8t)\left[(3t^2-4)\hat{\imath}+4t\hat{\jmath}\right]}{(9t^4-8t^2+16)^{-3/2}} + \frac{6t\hat{\imath}+4\hat{\jmath}}{(9t^4-8t^2+16)^{1/2}} \frac{(9t^4-8t^2+16)}{9t^4-8t^2+16}$$

$$= \Big[(-\cancel{54t^5} + 72t^3 + 24t^3 - 32t)\hat{\imath} + (-72t^4 + \cancel{32t^2})\hat{\jmath} + (\cancel{54t^5} - 48t^3 + 96t)\hat{\imath} + (36t^4 - \cancel{32t^2} + 64)\hat{\jmath} \Big] \Big/ (9t^4-8t^2+16)^{3/2}$$

$$= \frac{(48t^3+64t)\hat{\imath} + (-36t^4+64)\hat{\jmath}}{(9t^4-8t^2+16)^{3/2}} = \frac{4}{(9t^4-8t^2+16)^{3/2}}\left[4t(3t^2+4)\hat{\imath} - (9t^3-16)\hat{\jmath}\right]$$

$$\|T'(t)\| = \frac{4}{(9t^4-8t^2+16)^{3/2}}\sqrt{16t^2(3t^2+4)^2 + (9t^3-16)^2} =$$

$$\sqrt{16t^4(9t^4+24t^2+16) + 81t^6 - 288t^3 + 256}$$

$$\sqrt{144t^8 + 384t^6 + 256t^4 + 81t^6 - 288t^3 + 256} =$$

$$\sqrt{144t^8 + 465t^6 + 256t^4 - 288t^3 + 256}$$

$$N(t) = \frac{T'(t)}{\|T'(t)\|} = \frac{\cancel{4}}{\cancel{(9t^4-8t^2+16)^{3/2}}}\left[4t(3t^2+4)\hat{\imath} - (9t^3-16)\hat{\jmath}\right] \cdot \frac{\cancel{(9t^4-8t^2+16)^{3/2}}}{\cancel{4}} \cdot \frac{1}{\sqrt{144t^8+465t^6+256t^4-288t^3+256}}$$

$$= \frac{4t(3t^2+4)\hat{\imath} - (9t^3-16)\hat{\jmath}}{\sqrt{144t^8+465t^6+256t^4-288t^3+256}} \qquad N(1) = \frac{4\hat{\imath}+1\hat{\jmath}}{\sqrt{17}}$$

* there must be a mistake here, but I can't find it. :(I'm only human!

2a. $x^2+4y^2+z^2-36=0$

$\nabla F = \langle 2x, 8y, 2z\rangle$

$P(2,-2,4)$

$\nabla F(2,-2,4) = \langle 4,-16,8\rangle$

$4(x-2)-16(y+2)+8(z-4)=0$

$\vec{n}(t) = (4t+2)\hat{\imath} + (-16t-2)\hat{\jmath} + (8t+4)\hat{k}$

b. $y\ln xz^2-2=0$

$\nabla F = \langle \frac{yz^2}{xz^2}, \ln xz^2, \frac{2xz}{xz^2}\rangle = \langle \frac{y}{x}, \ln xz^2, \frac{2}{z}\rangle$

$\nabla F(e,2,1) = \langle \frac{2}{e}, 1, 2\rangle$

$\frac{2}{e}(x-e)+(y-2)+2(z-1)=0$

$\vec{n}(t) = (\frac{2}{e}t+e)\hat{\imath} + (t+2)\hat{\jmath} + (2t+1)\hat{k}$

c. $2xy-z^3=0$

$\nabla F = \langle 2y, 2x, -3z^2\rangle$

$\nabla F(2,2,2) = \langle 4,4,-12\rangle$

$4(x-2)+4(y-2)-12(z-2)=0$

$\vec{n}(t) = (4t+2)\hat{\imath} + (4t+2)\hat{\jmath} + (-12t+2)\hat{k}$

d. $\vec{r}_u = -2\sin u\,\hat{\imath} + 0\hat{\jmath} + 2\cos u\,\hat{k}$

$\vec{r}_v = 0\,\hat{\imath} + 1\hat{\jmath} + 0\,\hat{k}$

$u=0,\ v=4$

$\vec{r}_u \times \vec{r}_v = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ -2\sin u & 0 & 2\cos u \\ 0 & 1 & 0 \end{vmatrix} =$

$(-2\cos u)\hat{\imath} - 0\hat{\jmath} + (-2\sin u)\hat{k}$

$\langle -2, 0, 0\rangle$

$-2(x-2)+0(y-4)+0(z-0)=0$

$-2(x-2)=0 \Rightarrow \boxed{x=2}$

$\vec{n}(t) = (-2t-2)\hat{\imath} - 4\hat{\jmath}$

e. $\vec{r}_u = -3\cos v\sin u\,\hat{\imath} + 3\cos v\cos u\,\hat{\jmath} + 0\hat{k} \Rightarrow \langle 0,3,0\rangle$

$u=0,\ v=0$

$\vec{r}_v = -3\sin v\cos u\,\hat{\imath} - 3\sin v\sin u\,\hat{\jmath} + 5\cos v\,\hat{k} \Rightarrow \langle 0,0,5\rangle$

$r_u \times r_v = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ 0 & 3 & 0 \\ 0 & 0 & 5 \end{vmatrix} = (15-0)\hat{\imath} - (0-0)\hat{\jmath} + (0-0)\hat{k} = 15\hat{\imath} = \langle 15,0,0\rangle$

$15(x-3)+0(y-0)+0(z-0)=0 \Rightarrow 15(x-3)=0$

$\boxed{x=3}$

$\vec{n}(t) = (15t+3)\hat{\imath}$

f. $x^2+y^2+z-9=0$

$\nabla F = \langle 2x, 2y, 1\rangle$

$\nabla F(1,2,4) = \langle 2,4,1\rangle$

$2(x-1)+4(y-2)+1(z-4)=0$

$\vec{n}(t) = (2t+1)\hat{\imath} + (4t+2)\hat{\jmath} + (t+4)\hat{k}$

2g. $xyz - 10 = 0$

$\nabla F = \langle yz, xz, xy \rangle$

$\nabla F(1,2,5) = \langle 10, 5, 2 \rangle$

$10(x-1) + 5(y-2) + 2(z-5) = 0$ $\quad \vec{n}(t) = (10t+1)\hat{i} + (5t+2)\hat{j} + (2t+5)\hat{k}$

h. $\vec{r}_u = 1\hat{i} + 0\hat{j} + v\hat{k}$ $\quad u=1$

$\vec{r}_v = 0\hat{i} + 1\hat{j} + u\hat{k}$ $\quad v=1$

$r_u \times r_v = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{vmatrix} =$

$-1(x-1) - 1(y-1) + (z-1) = 0$ $\quad (0-1)\hat{i} - (1-0)\hat{j} + (1-0)\hat{k}$

$\vec{n}(t) = (1-t)\hat{i} + (1-t)\hat{j} + (t+1)\hat{k}$ $\quad \langle -1, -1, 1 \rangle$

i. $r_u = 1\hat{i} + 0\hat{j} + 0\hat{k}$ $\quad u = 1, v = 2$

$r_v = 0\hat{i} + \frac{3}{4}v^2\hat{j} + 1\hat{k}$

$r_u \times r_v = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 0 & 0 \\ 0 & 3 & 1 \end{vmatrix} =$

$0(x+1) + (-1)(y-2) + 3(z-2) = 0$ $\quad (0-0)\hat{i} - (1-0)\hat{j} + (3-0)\hat{k}$

$-(y-2) + 3(z-2) = 0$ $\quad \langle 0, -1, 3 \rangle$

$\vec{n}(t) = -\hat{i} + (2-t)\hat{j} + (3t+2)\hat{k}$

3. $r'(t) = \hat{i} + \frac{1}{2}(25-t^2)^{-1/2}(-2t)\hat{j} + \frac{1}{2}(25-t^2)^{-1/2}(-2t)\hat{k}$

$\hat{i} + \frac{-t}{\sqrt{25-t^2}}\hat{j} - \frac{t}{\sqrt{25-t^2}}\hat{k}$ $\quad t = 3$

$25 - 9 = 16$

$\Rightarrow \hat{i} - \frac{3}{4}\hat{j} - \frac{3}{4}\hat{k}$ $\quad \langle 1, -\frac{3}{4}, -\frac{3}{4} \rangle$

$r(3) = 3\hat{i} + 4\hat{j} + 4\hat{k}$

$L(t) = (t+3)\hat{i} + (\frac{3}{4}t + 4)\hat{j} + (\frac{3}{4}t + 4)\hat{k}$ $\quad \Delta t = .1$

$L(.1) = 3.1\hat{i} + 4.075\hat{j} + 4.075\hat{k}$

4. $r(t) = \cos 2t\,\hat{i} + \sin 2t\,\hat{j} + \frac{1}{\pi}t\hat{k}$

$r'(t) = -2\sin 2t\,\hat{i} + 2\cos 2t\,\hat{j} + \frac{1}{\pi}\hat{k}$

$\|r'(t)\| = \sqrt{4\sin^2 2t + 4\cos^2 2t + \frac{1}{\pi^2}} = \sqrt{4 + \frac{1}{\pi^2}}$

$$T(t) = \frac{-2\sin 2t\,\hat{i} + 2\cos 2t\,\hat{j} + \frac{1}{\pi}\hat{k}}{\sqrt{4 + \frac{1}{\pi^2}}}$$

4 cont'd

$$T'(t) = \frac{-4\cos 2t\,\hat{\imath} - 4\sin 2t\,\hat{\jmath} + 0\hat{k}}{\sqrt{4+\frac{1}{\pi^2}}}$$

$$\|T'(t)\| = \frac{1}{\sqrt{4+\frac{1}{\pi^2}}}\sqrt{16\cos^2 2t + 16\sin^2 2t}$$

$$N(t) = \frac{-\cancel{4}\cos 2t\,\hat{\imath} - 4\sin 2t\,\hat{\jmath} + 0\hat{k}}{\cancel{\sqrt{4+\frac{1}{\pi^2}}}} \cdot \frac{\cancel{\sqrt{4+\frac{1}{\pi^2}}}}{\cancel{4}} = -\cos 2t\,\hat{\imath} - \sin 2t\,\hat{\jmath}$$

$$T \times N = \frac{1}{\sqrt{4+\frac{1}{\pi^2}}}\begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ -2\sin 2t & 2\cos 2t & \frac{1}{\pi} \\ -\cos 2t & -\sin 2t & 0 \end{vmatrix} = \left(0 + \frac{\sin 2t}{\pi}\right)\hat{\imath} - \left(0 + \frac{\cos 2t}{\pi}\right)\hat{\jmath} + \left(2\sin^2 2t + 2\cos^2 2t\right)\hat{k}$$

$$\frac{\frac{\sin 2t}{\pi}\hat{\imath} - \frac{\cos 2t}{\pi}\hat{\jmath} + 2\hat{k}}{\sqrt{4+\frac{1}{\pi^2}}} = \vec{B}(t)$$

5. a.

$z = 1 + x\ln(xy-5)$ $\quad F = z - 1 - x\ln(xy-5)$

$\nabla F = \left\langle \ln(xy-5) - \frac{xy}{xy-5}, \frac{-x^2}{xy-5}, 1 \right\rangle$ $\quad \nabla F(2,3) = \langle -6, -4, 1 \rangle$

$z = 1$

$-6(x-2) - 4(y-3) + (z-1) = 0$

b. $z = x^2 + xz + 3y^2$ $\quad (1,1,2)$

$z - x^2 - xz - 3y^2 = F$

$\nabla F = \langle -2x - z, -6y, 1 - x \rangle$ $\quad \nabla F(1,1,2) = \langle -4, -6, 0 \rangle$

$-4(x-1) - 6(y-1) + 0(z-2) = 0$

c. $\vec{r}(u,v) = \sin u\,\hat{\imath} + \cos u \sin v\,\hat{\jmath} + \sin v\,\hat{k}$ $\quad (1,0,1)$ $\quad u = \pi/2$, $v = \pi/2$

$r_u = \cos u\,\hat{\imath} - \sin u \sin v\,\hat{\jmath} + 0\hat{k} \Rightarrow \langle 0, -1, 0 \rangle$

$r_v = 0\hat{\imath} + \cos u \cos v\,\hat{\jmath} + \cos v\,\hat{k} \Rightarrow \langle 0, 0, 0 \rangle$

$\vec{r}_u \times \vec{r}_v = \vec{0}$

no tangent plane at point

6. a. $\nabla f = \langle 3x^2, -3y^2 \rangle$ $\quad \nabla f(4,3) = \langle 48, -27 \rangle$

$\nabla f \cdot \vec{v} = \langle 48, -27 \rangle \cdot \frac{\sqrt{2}}{2}\langle 1, 1 \rangle = \frac{48\sqrt{2}}{2} - \frac{27\sqrt{2}}{2} = \frac{21\sqrt{2}}{2}$

b. $\nabla f = \langle y+z, x+z, y+x \rangle$ $\quad \nabla f(1,1,1) = \langle 2, 2, 2 \rangle$

$\hat{v} = \frac{2\hat{\imath} + \hat{\jmath} - \hat{k}}{\sqrt{6}}$ $\quad \nabla f \cdot \hat{v} = \langle 2,2,2 \rangle \cdot \frac{\langle 2, 1, -1 \rangle}{\sqrt{6}} = \frac{4}{\sqrt{6}} + \frac{2}{\sqrt{6}} - \frac{2}{\sqrt{6}} = \frac{4}{\sqrt{6}}$

⑥

6c. $\nabla f = \left\langle \frac{-y}{(x+y)^2}, \frac{1(x+y)-1(y)}{(x+y)^2} \right\rangle = \left\langle \frac{-y}{(x+y)^2}, \frac{x}{(x+y)^2} \right\rangle$

$\hat{v} = \left\langle \frac{\sqrt{3}}{2}, -\frac{1}{2} \right\rangle \qquad \nabla f \cdot \hat{v} = \frac{-\sqrt{3}y}{2(x+y)^2} - \frac{\frac{1}{2}x}{(x+y)^2} = \frac{-\sqrt{3}y - x}{2(x+y)^2}$

d. $\nabla f = \langle ye^z, xe^z, xye^z \rangle \qquad \nabla f(2,4,0) = \langle 4,2,8 \rangle$

$\vec{v} = \langle -2,-4,0 \rangle \qquad \hat{v} = \frac{\langle -2,-4,0 \rangle}{\sqrt{4+16}} = \frac{\langle -1,-2,0 \rangle}{\sqrt{5}}$

$= \sqrt{20} = 2\sqrt{5}$

$\nabla f \cdot \hat{v} = \langle 4,2,8 \rangle \cdot \frac{\langle -1,-2,0 \rangle}{\sqrt{5}} = \frac{-4-4+0}{\sqrt{5}} = \frac{-8}{\sqrt{5}}$

e. $\nabla f = \left\langle \frac{-y}{\sqrt{1-x^2y^2}}, \frac{-x}{\sqrt{1-x^2y^2}} \right\rangle \qquad \nabla f(1,0) = \langle 0,-1 \rangle$

$\hat{v} = \frac{\langle 1,5 \rangle}{\sqrt{26}} \qquad \nabla f \cdot \hat{v} = \langle 0,-1 \rangle \cdot \frac{\langle 1,5 \rangle}{\sqrt{26}} = 0 - \frac{5}{\sqrt{26}} = \frac{-5}{\sqrt{26}}$

f. $\nabla f = \langle 2\cos 2x \cos y, -\sin 2x \sin y \rangle \qquad \nabla f(0,0) = \langle 2,0 \rangle$

$\vec{v} = \left\langle \frac{\pi}{2}, \pi \right\rangle \qquad \hat{v} = \frac{\langle \frac{\pi}{2}, \pi \rangle}{\sqrt{\frac{\pi^2}{4}+\pi^2}} = \frac{\langle \pi/2, \pi \rangle}{\pi\sqrt{5/4}} = \frac{\langle 1/2, 1 \rangle}{\sqrt{5/4}} = \frac{\langle 1,2 \rangle}{\sqrt{5}}$

$\nabla f \cdot \hat{v} = \langle 2,0 \rangle \cdot \frac{\langle 1,2 \rangle}{\sqrt{5}} = \frac{2}{\sqrt{5}}$

7. The gradient for directional derivatives is gradient of an explicit function. The gradient for normal vector to a surface is one higher variable to incorporate change in all physical directions new function made w/all variables on one side of equation.

8. a. $F = x^2 + y^2 - z \qquad \nabla F = \langle 2x, 2y, -1 \rangle \qquad \nabla F(2,-1,5) = \langle 4,-2,-1 \rangle$

$G = 4 - y - z \qquad \nabla G = \langle 0,-1,-1 \rangle$

$\langle 4,-2,-1 \rangle \cdot \langle 0,-1,-1 \rangle = 0+2+1 = 3$

$\|\nabla F\| = \sqrt{16+4+1} = \sqrt{21} \qquad \|\nabla G\| = \sqrt{2}$

$\cos^{-1}\left(\frac{3}{\sqrt{21}\sqrt{2}}\right) = 1.09$ radians

$\approx 62.4^\circ$ $\boxed{\text{no}}$

b. $F = x^2 + z^2 - 25 \qquad \nabla F = \langle 2x, 0, 2z \rangle \qquad \nabla F(3,3,4) = \langle 6,0,8 \rangle$

$G = y^2 + z^2 - 25 \qquad \nabla G = \langle 0, 2y, 2z \rangle \qquad \nabla G(3,3,4) = \langle 0,6,8 \rangle$

$\langle 6,0,8 \rangle \cdot \langle 0,6,8 \rangle = 0+0+64$

$\|\nabla F\| = \sqrt{36+64} = 10 \qquad \|\nabla G\| = \sqrt{36+64} = 10$

$\cos^{-1}\left(\frac{64}{100}\right) = 50.2^\circ$

$\approx .876$ radians

$\boxed{\text{no}}$

9. $\nabla f = \langle -ye^{-x}, e^{-x} \rangle \quad \nabla f(0,4) = \langle -4, 1 \rangle$

$\hat{v} = \dfrac{\langle 3, 5 \rangle}{\sqrt{9+25}} = \dfrac{\langle 3, 5 \rangle}{\sqrt{34}}$

$\nabla f \cdot \hat{v} = \langle -4, 1 \rangle \cdot \dfrac{\langle 3, 5 \rangle}{\sqrt{34}} = \dfrac{-12+5}{\sqrt{34}} = \dfrac{-7}{\sqrt{34}}$

direction of maximum $\langle -4, 1 \rangle$ or $\dfrac{\langle -4, 1 \rangle}{\sqrt{17}} = \nabla f$ or $\dfrac{\nabla f}{\|\nabla f\|}$