277 Homework #8 Key

1a. $r'(t) = 1\hat{i} + 2t\hat{j}$ $\quad \int_0^4 \sqrt{1+4t}\,dt \approx 16.8186$

b. $r'(t) = 0\hat{i} + 2t\hat{j} + 3t^2\hat{k}$ $\quad \int_0^2 \sqrt{4t^2+9t^4}\,dt \approx 9.0734$

c. $r'(t) = -a\sin t\,\hat{i} + a\cos t\,\hat{j}$ $\quad \int_0^{2\pi} \sqrt{a^2\sin^2 t + a^2\cos^2 t}\,dt = \int_0^{2\pi} \sqrt{a^2}\,dt =$

$\int_0^{2\pi} a\,dt = 2\pi a$

d. $r'(t) = (-\cancel{\sin t} + \cancel{\sin t} + t\cos t)\hat{i} + (\cancel{\cos t} - \cancel{\cos t} + t\sin t)\hat{j} + 2t\hat{k}$

$\int_0^{\pi/2} \sqrt{t^2\cos^2 t + t^2\sin^2 t + 4t^2}\,dt = \int_0^{\pi/2} \sqrt{t^2 + 4t^2}\,dt = \int_0^{\pi/2} \sqrt{5t^2}\,dt =$

$\int_0^{\pi/2} \sqrt{5}\,t\,dt = \frac{\sqrt{5}}{2}t^2\Big|_0^{\pi/2} = \frac{\sqrt{5}}{2}\left(\frac{\pi}{2}\right)^2 = \frac{\sqrt{5}\pi^2}{8}$

2.a. $r'(s) = 1\hat{i} + 0\hat{j}$

$\|r''(s)\| = K = 0$ (straight line, no curvature) $\quad R = \frac{1}{0} = \infty$

b. $r'(t) = 4(\cancel{\cos t} - \cancel{\cos t} + t\sin t)\hat{i} + 4(\cancel{\sin t} + \cancel{\sin t} + t\cos t)\hat{j} + \frac{4}{3}t\,\hat{k}$

$r''(t) = 4(\sin t + t\cos t)\hat{i} + 4(\cos t - t\sin t)\hat{j} + \frac{4}{3}\hat{k}$

$$r' \times r'' = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 4t\sin t & 4t\cos t & 4/3\,t \\ 4(\sin t + t\cos t) & 4(\cos t - t\sin t & 4/3 \end{vmatrix} =$$

$$\left(\tfrac{16}{3}\cancel{t\cos t} - \tfrac{16}{3}\cancel{t\cos t} + \tfrac{16}{3}t^2\sin t\right)\hat{i} - \left(\tfrac{16}{3}\cancel{t\sin t} - \tfrac{16}{3}\cancel{t\sin t} - \tfrac{16}{3}t^2\cos t\right)\hat{j} +$$
$$\left(16t\cancel{\cos t\sin t} - \underbrace{16t^2\sin^2 t - 16t\cancel{\cos t\sin t} - 16t^2\cos^2 t}_{=1}\right)\hat{k}$$

$= \left(\frac{16}{3}t^2\sin t\right)\hat{i} + \left(\frac{16}{3}t^2\cos t\right)\hat{j} - 16t^2\hat{k}$

$\|r' \times r''\| = \sqrt{\underbrace{\frac{256}{9}t^4\sin^2 t + \frac{256}{9}t^4\cos^2 t}_{=1} + 256t^4} = \sqrt{\frac{256}{9}t^4 + 256t^4} = t^2\sqrt{\frac{2560}{9}}$

$= \frac{16\sqrt{10}}{3}t^2$

$\|r'(t)\| = \sqrt{\underbrace{16t^2\sin^2 t + 16t^2\cos^2 t}_{=1} + \frac{16}{9}t^2} = \sqrt{16t^2 + \frac{16}{9}t^2} = \sqrt{\frac{160}{9}t^2} = \frac{4\sqrt{10}}{3}t$

$K = \dfrac{\cancel{\frac{16\sqrt{10}}{3}}\cancel{t^2}}{\cancel{4^3 \cdot 10\sqrt{10}/27\,t^3}} = \dfrac{9}{40t} \quad K(0) = \infty$

2c. $r'(t) = -5\sin t\,\hat{i} - 4\sin t\,\hat{j}$ $\quad t = \frac{\pi}{3}$

$r''(t) = -5\cos t\,\hat{i} - 4\cos t\,\hat{j}$

$r'(t) \times r''(t) = \begin{vmatrix} \hat{i} & \hat{j} & \hat{j} \\ -5\sin t & -4\sin t & 0 \\ -5\cos t & -4\cos t & 0 \end{vmatrix} = (0)\hat{i} + (0)\hat{j} + (20\sin t\cos t - 20\sin t\cos t)\hat{k}$
$= \vec{0}$

Straight line so $K = 0$ $\quad R = \infty$ everywhere

d. $r'(s) = -\frac{1}{\sqrt{2}}\hat{i} + \frac{1}{\sqrt{2}}\hat{j}$ $\quad s = 1$

$r''(s) = 0\hat{i} + 0\hat{j}$ $\quad \|r''(s)\| = 0$ $\quad K = 0$ $\quad \frac{1}{K} = R = \infty$

straight line

e. $r'(t) = (e^t\cos t - e^t\sin t)\hat{i} + (e^t\sin t + e^t\cos t)\hat{j} + e^t\hat{k}$ $\quad t = \pi$

$r''(t) = (\cancel{e^t\cos t} - e^t\sin t - e^t\sin t - \cancel{e^t\cos t})\hat{i} + (\cancel{e^t\sin t} + e^t\cos t +$
$e^t\cos t - \cancel{e^t\sin t})\hat{j} + e^t\hat{k} =$
$(-2e^t\sin t)\hat{i} + 2e^t\cos t\,\hat{j} + e^t\hat{k}$

$r'(\pi) = -e^{\pi}\hat{i} + -e^{\pi}\hat{j} + e^{\pi}\hat{k}$ $\quad r''(\pi) = 0\hat{i} + -2e^{\pi}\hat{j} + e^{\pi}\hat{k}$

$r' \times r'' = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ -e^{\pi} & -e^{\pi} & e^{\pi} \\ 0 & -2e^{\pi} & e^{\pi} \end{vmatrix} = (-e^{2\pi} + 2e^{2\pi})\hat{i} - (-e^{2\pi} - 0)\hat{j} + (2e^{2\pi} - 0)\hat{k}$
$e^{2\pi}\hat{i} + e^{2\pi}\hat{j} + 2e^{2\pi}\hat{k}$

$\|r' \times r''\| = \sqrt{e^{4\pi} + e^{4\pi} + 4e^{4\pi}} = \sqrt{6e^{4\pi}} = \sqrt{6}\,e^{2\pi}$

$\|r'(\pi)\| = \sqrt{e^{2\pi} + e^{2\pi} + e^{2\pi}} = \sqrt{3e^{2\pi}} = \sqrt{3}\,e^{\pi}$

$K = \frac{\overset{\sqrt{2}}{\cancel{\sqrt{6}}}\,e^{2\pi}}{3\cancel{\sqrt{3}}\,e^{3\pi}} = \frac{\sqrt{2}}{3e^{\pi}}$ $\quad R = \frac{1}{K} = \frac{3e^{\pi}}{\sqrt{2}}$

3a. $r'(t) = 2t\,\hat{i} + \frac{1}{t}\hat{j} + (\ln t + 1)\hat{k}$ $\quad (1,0,0) \Rightarrow t = 1$

$r''(t) = 2\hat{i} - \frac{1}{t^2}\hat{j} + \frac{1}{t}\hat{k}$ $\quad r'(1) = 2\hat{i} + 1\hat{j} + 1\hat{k}$
$r''(1) = 2\hat{i} - 1\hat{j} + 1\hat{k}$

$r' \times r'' = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2 & 1 & 1 \\ 2 & -1 & 1 \end{vmatrix} = (1+1)\hat{i} - (2-2)\hat{j} + (-2-2)\hat{k}$
$\langle 2, 0, -4 \rangle$

$\|r' \times r''\| = \sqrt{4 + 16} = \sqrt{20} = 2\sqrt{5}$

$\|r'(t)\| = \sqrt{4+1+1} = \sqrt{6}$

$K = \frac{\cancel{2}\sqrt{5}}{3\cancel{6}\sqrt{6}} = \frac{\sqrt{5}}{3\sqrt{6}}$

$R = \frac{3\sqrt{6}}{\sqrt{5}}$

Sketch curve w/ technology

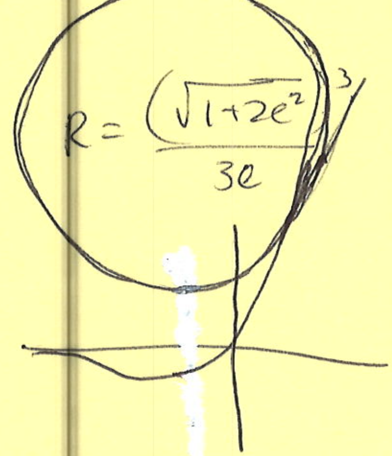
3b. $f'(x) = e^x + xe^x = (x+1)e^x$

$f''(x) = e^x + (x+1)e^x = (x+2)e^x$

$$k = \frac{(x+2)e^x}{\left(\sqrt{1+(x+1)^2e^{2x}}\right)^3}, \quad K(1) = \frac{3e}{\left(\sqrt{1+2e^2}\right)^3}$$

$x = 1$

$$R = \frac{\left(\sqrt{1+2e^2}\right)^3}{3e}$$



4a.

$z = \sqrt{36 - x^2 - y^2}$ (top hemisphere)

$f = z - \sqrt{36-x^2-y^2}$ $\quad \nabla f = \left\langle \frac{x}{\sqrt{36-x^2-y^2}}, \frac{y}{\sqrt{36-x^2-y^2}}, 1 \right\rangle$ upward normal outward here

b. $F = z - 1 + x^2 + y^2$ $\quad \nabla F = \langle 2x, 2y, 1\rangle$ upward = outward

c. $F = 3x + 2y + z - 6 = 0$ $\quad \nabla F = \langle 3, 2, 1\rangle$ upward = outward (plane so no "outward" or "inward" w/o info about coordinate planes)

d. $z = \sqrt{36-x^2-y^2}$ Same as a except no second hemisphere

e. $f = z - x^2 - y^2$ $\quad \nabla f = \langle -2x, -2y, 1\rangle$ upward = outward

5a. $z = 12 + 2x - 3y$ $\quad G = 12 + 2x - 3y - z$ $\quad \nabla G = \langle 2, -3, -1\rangle$

$\|\nabla G\| = \sqrt{4+9+1} = \sqrt{14}$ $\quad \int_0^{2\pi}\int_0^3 \sqrt{14}\, r\,dr\,d\theta = \sqrt{14}\cdot\pi(3)^2 = 9\sqrt{14}\,\pi$

b. $z = 3 + x^{3/2}$ $\quad G = 3 + x^{3/2} - z$ $\quad \nabla G = \langle \frac{3}{2}x^{1/2}, 0, -1\rangle$

$\|\nabla G\| = \sqrt{\frac{9}{4}x + 1}$ $\quad \int_0^3\int_0^4 \sqrt{\frac{9}{4}x+1}\,dy\,dx = 4\int_0^3 \sqrt{\frac{9}{4}x+1}\,dx \approx 24.385$

c. $z = \ln|\sec x|$ $\quad G = \ln|\sec x| - z$ $\quad \nabla G = \langle \tan x, 0, -1\rangle$

$\|\nabla G\| = \sqrt{\tan^2 x + 1} = \sqrt{\sec^2 x} = |\sec x|$

$$\int_0^{\pi/4}\int_0^{\tan x} |\sec x|\,dy\,dx = \int_0^{\pi/4} \sec x \tan x\,dx = \sec x \Big|_0^{\pi/4} = \boxed{\sqrt{2} - 1}$$

5d. $z=\sqrt{x^2+y^2}$ $\quad G=\sqrt{x^2+y^2}-z$ $\quad \nabla G=\left\langle \frac{\cancel{2}x}{\cancel{2}\sqrt{x^2+y^2}}, \frac{y}{\sqrt{x^2+y^2}}, -1\right\rangle$

$$\|\nabla G\|=\sqrt{\frac{x^2}{x^2+y^2}+\frac{y^2}{x^2+y^2}+1}=\sqrt{\frac{x^2+y^2}{x^2+y^2}+1}=\sqrt{1+1}=\sqrt{2}$$

$$\int_0^{2\pi}\int_0^3 \sqrt{2}\, r\,dr\,d\theta=\sqrt{2}\cdot\pi(3)^2=9\sqrt{2}\pi$$

e. $r_u=4\hat{i}-0\hat{j}+0\hat{k}$ $\quad r_u\times r_v=\begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ 4 & 0 & 0\\ 0 & -1 & 1\end{vmatrix}=(0)\hat{i}-4\hat{j}-4\hat{k}$

$r_v=0\hat{i}-1\hat{j}+1\hat{k}$ $\quad \langle 0,-4,-4\rangle$

$$\|\vec{r_u}\times\vec{r_v}\|=\sqrt{16+16}=\sqrt{32}=4\sqrt{2}$$

$$\int_0^1\int_0^2 4\sqrt{2}\,du\,dv=\int_0^1 8\sqrt{2}\,dv=8\sqrt{2}$$

f. $r_u=2\cos v\,\hat{i}+2\sin v\,\hat{j}+2u\,\hat{k}$ $\quad \vec{r_u}\times\vec{r_v}=\begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ 2\cos v & 2\sin v & 2u\\ -2u\sin v & 2u\cos v & 0\end{vmatrix}$

$r_v=-2u\sin v\,\hat{i}+2u\cos v\,\hat{j}+0\hat{k}$

$$=(-4u^2\cos v)\hat{i}-(+4u^2\sin v)\hat{j}+(4u\underbrace{\cos^2 v+4u\sin^2 v}_{=1})\hat{k}$$

$$=-4u^2\cos v\,\hat{i}-4u^2\sin v\,\hat{j}+4u\,\hat{k}$$

$$\|r_u\times r_v\|=\sqrt{16u^4\underbrace{\cos^2 v+16u^4\sin^2 v}_{=1}+16u^2}=\sqrt{16u^2(u^2+1)}=4u\sqrt{u^2+1}$$

$$\int_0^{2\pi}\int_0^2 4u\sqrt{u^2+1}\,du\,dv=$$

6. ∇f a, b, e, g

∇F c, f

we don't have case a, b, e, g for parametric surfaces in our text/course

only c, f w/ $r_u\times r_v$.