277 Homework #9 Key

1a. $\int_C \underbrace{2xy}_{M}\,dx + \underbrace{(x+y)}_{N}\,dy$ $\quad y=0,\ y=1-x^2$

$\frac{\partial N}{\partial x} = 1 \quad \frac{\partial M}{\partial y} = 2x$

$\int_{-1}^{1}\int_{0}^{1-x^2} 1-2x\ dy\,dx = \int_{-1}^{1} (1-2x)y\Big|_0^{1-x^2} dx =$

$\int_{-1}^{1}(1-2x)(1-x^2)\,dx = \int_{-1}^{1} 1-2x-x^2+2x^3\,dx = \int_{-1}^{1} 1-x^2\,dx + \int_{-1}^{1} -2x+2x^3\,dx$

$\Rightarrow 2\int_0^1 1-x^2\,dx$ even $\qquad$ odd $\Rightarrow 0$

$= 2\left(x-\frac{1}{3}x^3\right)\Big|_0^1 = 2\left(1-\frac{1}{3}\right) = 2\left(\frac{2}{3}\right) = \boxed{\frac{4}{3}}$

b. $\int_C \underbrace{\cos y}_{M}\,dx + \underbrace{(xy + x\sin y)}_{N}\,dy$ $\quad y = x,\ y=\sqrt{x}$

$\frac{\partial N}{\partial x} = y + \sin y \quad \frac{\partial M}{\partial y} = -\sin y$

$\int_0^1\int_x^{\sqrt{x}} (y+\sin y)-(-\sin y)\,dy\,dx = \int_0^1\int_x^{\sqrt{x}} 2\sin y + y\,dy\,dx = \int_0^1 -2\cos y + \frac{1}{2}y^2\Big|_x^{\sqrt{x}} dx$

$= \int_0^1 -2\cos\sqrt{x} + 2\cos x + \frac{1}{2}x - \frac{1}{2}x^2\,dx \approx .2392$

2a. $\vec{F}(x,y,z) = x^2\hat{\imath} + z^2\hat{\jmath} - xyz\hat{k}$, $\quad S: z = \sqrt{4-x^2-y^2}$

hemisphere

$\nabla\times F = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ x^2 & z^2 & -xyz \end{vmatrix} = (-xz-2z)\hat{\imath} - (-yz-0)\hat{\jmath} + (0-0)\hat{k}$

Surface of base is xy-plane, $n = \hat{k}$

$(\nabla\times F)\cdot\hat{n} = \langle -xz-2z, yz, 0\rangle \cdot \langle 0,0,1\rangle = 0$

$\int_C \vec{F}\cdot d\vec{r} = \iint_S (\nabla\times F)\cdot N\,dS = \int_0^{2\pi}\int_0^2 0\,dA = 0$

b. $F(x,y,z) = yz\hat{\imath} + (2-3y)\hat{\jmath} + (x^2+y^2)\hat{k}$ $\quad$ S: first octant of $x^2+z^2=16$ over $x^2+y^2=16$

$\nabla\times F = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ yz & 2-3y & x^2+y^2 \end{vmatrix} =$

$(2y-0)\hat{\imath} - (2\cancel{x}-y)\hat{\jmath} + (0-z)\hat{k}$

$z = \sqrt{16-x^2}$

$G = z - \sqrt{16-x^2}$

$\nabla G = \left\langle \frac{x}{\sqrt{16-x^2}}, 0, 1\right\rangle$

$(\nabla\times F)\cdot\nabla G = \langle 2y, y-2x, -z\rangle \cdot \left\langle \frac{x}{\sqrt{16-x^2}}, 0, 1\right\rangle = \frac{2xy}{\sqrt{16-x^2}} + 0 - z = \frac{2xy}{\sqrt{16-x^2}} - \sqrt{16-x^2}$

2b cont'd

$$\int_0^{2\pi}\int_0^4 \left[\frac{2r\cos\theta\, r\sin\theta}{\sqrt{16-r^2\cos^2\theta}} - \sqrt{16-r^2\cos^2\theta}\right] r\,dr\,d\theta$$ nasty!

use $\hat{k}=\hat{n}$

$(\nabla\times F)(\hat{n}) = \langle 2y,\ y-2x,\ -z\rangle\cdot\langle 0,0,1\rangle = -z = -\sqrt{16-r^2\cos\theta}$

$= -\sqrt{16-x^2}$

$\int_0^{2\pi}\int_0^4 -\sqrt{16-r^2\cos^2\theta}\cdot r\,dr\,d\theta$

$u=16-r^2\cos^2\theta$

$du=-2r\cos^2\theta\,dr$

$-\frac{1}{2}\sec^2\theta\,du = r\,dr$

$\int +\frac{1}{2}\sec^2\theta\, u^{1/2}\,du \Rightarrow$

$\frac{1}{\cancel{2}}\sec^2\theta\,\frac{\cancel{2}}{3}u^{3/2}\Big| = \frac{1}{3}\sec^2\theta\,(16-r^2\cos^2\theta)^{3/2}\Big|_0^4 = \frac{1}{3}\sec^2\theta\,(16-16\cos^2\theta)^{3/2} - \frac{1}{3}\sec^2\theta\,(16)^{3/2}$

$(\sqrt{16})^3 = 4^3 = 64$

$\int_0^{2\pi}\frac{1}{3}\sec^2\theta\left[64\sin^3\theta-64\right]d\theta =$

$16-16\cos^2\theta =$

$16(1-\cos^2\theta)\Rightarrow$

$(16\sin^2\theta)^{3/2} = 64\sin^3\theta$

$\frac{64}{3}\int_0^{2\pi}\sec^2\theta\,(\sin^3\theta-1)\,d\theta =$

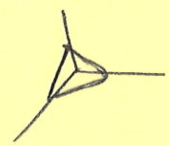
use symmetry $\frac{64}{3}\cdot 4\int_0^{2\pi}\sec^2\theta\,(\sin^3\theta-1)\,d\theta = \frac{64}{3}\cdot 4(-2) = -\frac{512}{3}$

c. $F(x,y,z) = (x+y^2)\hat{i} + (y+z^2)\hat{j} + (xy-\sqrt{z})\hat{k}$

S: plane: $x+y+z=1$

$\nabla G = \langle 1,1,1\rangle$

$$\begin{array}{c} (1,0,0) \\ (0,1,0) \\ \hline \langle 1,-1,0\rangle \end{array} \qquad \begin{array}{c} (0,1,0) \\ (0,0,1) \\ \hline \langle 0,1,-1\rangle \end{array}$$



$$\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & -1 & 0 \\ 0 & 1 & -1 \end{vmatrix} =$$

$(1-0)\hat{i} - (-1-0)\hat{j} + (1-0)\hat{k}$

$= \langle 1,1,1\rangle$

$$\nabla\times F = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ x+y^2 & y+z^2 & xy-\sqrt{x} \end{vmatrix} =$$

$(x-2z)\hat{i} - (y-0)\hat{j} + (0-2y)\hat{k} = \langle x-2z,\ -y,\ -2y\rangle$

$(\nabla\times F)\cdot\nabla G = \langle x-2z,\ -y,\ -2y\rangle\langle 1,1,1\rangle = x-2z-y-2y = x-2z-3y$

$z = 1-x-y \longrightarrow \quad x-2(1-x-y)-3y = x-2+2x+2y = 3x+2y-2$

$\int_0^1\int_0^{1-x} 3x+2y-2\,dy\,dx = \int_0^1 3xy+y^2-2y\Big|_0^{1-x}dx = \int_0^1 3x(1-x)+(1-x)^2-2(1-x)\,dx$

$= \int_0^1 3x-3x^2+1-\cancel{2x}+x^2-2+\cancel{2x}\,dx = \int_0^1 -2x^2+3x-1\,dx = -\frac{2}{3}x^3+\frac{3}{2}x^2-x\Big|_0^1 =$

$-\frac{2}{3}+\frac{3}{2}-1 = \boxed{-\frac{1}{6}}$

3. $$\nabla\times F = \begin{vmatrix} \hat{i} & \hat{j} & \hat{j} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ yz & 2xz & e^{xy} \end{vmatrix} = (xe^{xy}-2x)\hat{i} - (ye^{xy}-y)\hat{j} + (2\cancel{z}-z)\hat{k}$$

$= \langle xe^{xy}-2x,\ y-ye^{xy},\ 5\rangle$

$\nabla G = \hat{k}$ 3 cont'd

$\langle xe^{xy}-2x,\ y-ye^{xy},\ 5\rangle\cdot\langle 0,0,1\rangle = 5 = (\nabla\times F)\cdot\nabla G$

$\int_0^{2\pi}\int_0^4 5\, r\,dr\,d\theta = \; = 5$ area of circle $= 5\cdot\pi(4)^2 = 80\pi$

OR

$r = 4\cos t\,\hat{\imath} + 4\sin t\,\hat{\jmath} + 5\hat{k}$

$r' = -4\sin t\,\hat{\imath} + 4\cos t\,\hat{\jmath} + 0\hat{k}$

$F\cdot dr = \langle yz,\ 2xz,\ e^{xy}\rangle\cdot\langle -4\sin t,\ 4\cos t,\ 0\rangle =$

$\langle 20\sin t,\ 40\cos t,\ e^{16\sin t\cos t}\rangle\cdot\langle -4\sin t,\ 4\cos t,\ 0\rangle =$

$-80\sin^2 t + 160\cos^2 t \quad = 80(\cos^2 t - \sin^2 t) = 80\cos 2t =$

$+80\cos^2 t \qquad 40(1+\cos 2t)$

$\int_0^{2\pi} 40 + 120\cos 2t\,dt = 40t + 60\sin 2t\Big|_0^{2\pi} = 80\pi + 0 = \boxed{80\pi}$

4a. $F = yz\,\hat{\imath} + xz\,\hat{\jmath} + xy\,\hat{k}$ $(0,0,0)$ to $(5,3,2)$

Conservative

$\int yz\,dx = xyz + f(y,z)$

$\int xz\,dy = xyz + g(x,z)$ $\qquad \varphi = (xyz)$

$\int xy\,dz = xyz + h(x,y)$ $\qquad \int_C \vec{F}\cdot d\vec{r} =$

$\varphi(5,3,2) - \varphi(0,0,0) = 5\cdot 3\cdot 2 - 0\cdot 0\cdot 0 = \boxed{30}$

Conservative

b. $F = (x^2+y^2)\,\hat{\imath} + 2xy\,\hat{\jmath}$

$\int x^2+y^2\,dx = \frac{1}{3}x^3 + xy^2 + f(y)$ $\qquad \varphi = \frac{1}{3}x^3 + xy^2$

$\int 2xy\,dy = xy^2 + g(x)$

$\varphi(8,4) - \varphi(0,0) = \frac{1}{3}(8)^3 + 8(4)^2 - 0 = \frac{896}{3}$

r_1: $(0,0)$ to $(8,4)$

$\varphi(0,2) - \varphi(2,0) = \frac{1}{3}(0)^3 + 0 - \frac{1}{3}(2)^3 - 0 = -\frac{8}{3}$

r_2: $(2,0)$ to $(0,2)$

5a. definition only

b. definition $\quad \int x+y\,dx = \frac{1}{2}x^2 + xy + f(y,z)$ $\quad$ definition only

$\int(y-z)\,dy = \frac{1}{2}y^2 - yz$ $\quad$ not conservative

c. definition, FTLI (better)

d. definition only $\quad$ not conservative

5e. definition, Green's theorem, (eq. to Stokes')

f. definition would require information on path (more than available)

FTLI → field conservative

g. definition, Stokes' best

h. definition, Green's theorem best

i. definition, Stokes' theorem best

6a. $G = \frac{2}{3}x^{3/2} - z \qquad \nabla G = \langle x^{1/2}, 0, -1 \rangle$

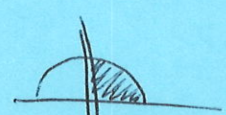
$\|\nabla G\| = \sqrt{x+1}$

$$\int_0^1\int_0^x (x-2y+z)\sqrt{x+1}\,dy\,dx = \int_0^1\int_0^x \left(x-2y+\tfrac{2}{3}x^{3/2}\right)\sqrt{x+1}\,dy\,dx$$

$$\int_0^1 \left(x+\tfrac{2}{3}x^{3/2}\right)\sqrt{x+1}\,y - y^2\sqrt{x+1}\,\Big|_0^x\,dx = \int_0^1 \left(\cancel{x^2}+\tfrac{2}{3}x^{5/2}\right)\sqrt{x+1} - \cancel{x^2}\sqrt{x+1}\,dx$$

$$= \int_0^1 \tfrac{2}{3}x^{5/2}\sqrt{x+1}\,dx \approx .2536$$

b. $G = \frac{1}{2}xy - z \qquad \nabla G = \langle \frac{1}{2}y, \frac{1}{2}x, -1 \rangle$



$\|\nabla G\| = \sqrt{\frac{1}{4}x^2+\frac{1}{4}y^2+1}$

$$\underbrace{\frac{1}{2}\int_0^{\pi/2}\int_0^2}_{\text{polar}} (xy)\sqrt{x^2+y^2+4}\,dA = \frac{1}{2}\int_0^{\pi/2}\int_0^2 r\cos\theta\, r\sin\theta\sqrt{r^2+4}\; r\,dr\,d\theta$$

$$= \frac{1}{2}\int_0^{\pi/2}\cos\theta\sin\theta\,d\theta \cdot \int_0^2 r^3\sqrt{r^2+4}\,dr \approx 2.575$$

c. $r_u = -2\sin u\,\hat{i} + 2\cos u\,\hat{j} + 0\hat{k} \qquad \vec{r}_u \times \vec{r}_v = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ -2\sin u & 2\cos u & 0 \\ 0 & 0 & 1 \end{vmatrix} =$

$r_v = 0\hat{i} + 0\hat{j} + 1\hat{k}$

$(2\cos u - 0)\hat{i} - (-2\sin u)\hat{j} + 0\hat{k}$

$\langle 2\cos u, 2\sin u, 0 \rangle$

$\|\vec{r}_u \times \vec{r}_v\| = \sqrt{4\cos^2 u + 4\sin^2 u} = \sqrt{4} = 2$

$$\int_0^{\pi/2}\int_0^1 (2\cos u + 2\sin u)2\,dv\,du =$$

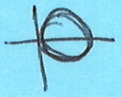
$$\int_0^{\pi/2} 4\cos u + 4\sin u\,du = 4\sin u - 4\cos v\Big|_0^{\pi/2}$$

$$(4-0)-(0-4) = \boxed{8}$$

d. $G = \sqrt{x^2+y^2} - z \qquad \nabla G = \left\langle \frac{x}{\sqrt{x^2+y^2}}, \frac{y}{\sqrt{x^2+y^2}}, -1 \right\rangle \qquad \|\nabla G\| = \sqrt{\frac{x^2}{x^2+y^2}+\frac{y^2}{x^2+y^2}+1} = \sqrt{\frac{x^2+y^2}{x^2+y^2}+1}$

$= \sqrt{1+1} = \sqrt{2}$

6d cont'd

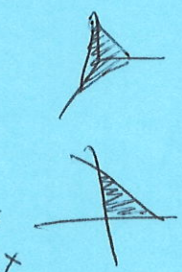
 $r = 2\cos\theta$

$f = \sqrt{x^2+y^2+z^2} = \sqrt{x^2+y^2+\left(\sqrt{x^2+y^2}\right)^2} = \sqrt{2x^2+2y^2} = \sqrt{2}\,r$

$\int_0^{\pi}\int_0^{2\cos\theta} \underbrace{\sqrt{2}r\cdot\sqrt{2}\,r}_{2r^2}\,dr\,d\theta = \int_0^{\pi} \frac{2}{3}r^3\Big|_0^{2\cos\theta} d\theta = \int_0^{\pi}\frac{2}{3}(8\cos^3\theta)\,d\theta$

$\frac{16}{3}\int_0^{\pi}\cos\theta(1-\sin^2\theta)\,d\theta$ $\quad u=\sin\theta$, $du=\cos\theta$ $\quad \int 1-u^2\,du = u-\frac{1}{3}u^3$

$\frac{16}{3}\left[\sin\theta - \frac{1}{3}\sin^3\theta\right]_0^{\pi} = \boxed{0}$



7a. $G = z+3x+2y-6=0$ $\quad \nabla G = \langle 3,2,1\rangle$

$F\cdot\nabla G = \langle x,y,0\rangle\cdot\langle 3,2,1\rangle = 3x+2y$

$z=0$

$2y = 6-3x$

$y = 3-\frac{3}{2}x$

$\int_0^2\int_0^{3-\frac{3}{2}x} 3x+2y\,dy\,dx$

$\int_0^2 3xy+y^2\Big|_0^{3-\frac{3}{2}x}dx = \int_0^2 3x(3-\frac{3}{2}x)+(3-\frac{3}{2}x)^2\,dy = \boxed{12}$

b. $G = z-1+x^2+y^2$ $\quad \nabla G = \langle 2x,2y,1\rangle$

$F\cdot\nabla G = \langle x,y,z\rangle\cdot\langle 2x,2y,1\rangle = 2x^2+2y^2+z = 2x^2+2y^2+(1-x^2-y^2) =$
x^2+y^2+1

$\int_0^{2\pi}\int_0^1 \underbrace{(r^2+1)r}_{r^3+r}\,dr\,d\theta = \int_0^{2\pi}\frac{1}{4}r^4+\frac{1}{2}r^2\Big|_0^1 d\theta = 2\pi\left(\frac{3}{4}\right) = \boxed{\frac{3\pi}{2}}$

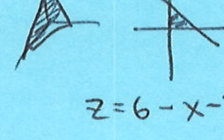
8a. $\nabla\cdot F = 1+1+1 = 3$ $\quad z = 16-r^2$

$\int_0^{2\pi}\int_0^4\int_0^{16-r^2} 3\,r\,dz\,dr\,d\theta = \int_0^{2\pi}\int_0^4 3(16r-r^3)\,dr\,d\theta = 3\int_0^{2\pi} 8r^2-\frac{1}{4}r^4\Big|_0^4 d\theta$

$= 3\int_0^{2\pi} 64\,d\theta = \boxed{384\pi}$

b. $\nabla\cdot F = 2-2+2z = 2z$

$\int_0^{2\pi}\int_0^2\int_0^5 2zr\,dz\,dr\,d\theta = \int_0^{2\pi}\int_0^2 z^2\Big|_0^5 r\,dr\,d\theta = \int_0^{2\pi}\int_0^2 25\,r\,dr\,d\theta = 25\pi(2)^2 = \boxed{100\pi}$



$2x+4y=12$
$x+2y=6$
$2y=6-x$
$y=3-\frac{1}{2}x$

c. $\nabla\cdot F = 2-2+1=1$

$\int_0^6\int_0^{3-\frac{1}{2}x}\int_0^{6-x-2y} 1\,dz\,dy\,dx = \int_0^6\int_0^{3-\frac{1}{2}x} 6-x-2y\,dy\,dx$

$z = 6-x-2y$

8c cont'd

$$\int_0^6 6y - xy - y^2 \Big|_0^{3-\frac{1}{2}x} dx = \int_0^6 6(3-\tfrac{1}{2}x) - x(3-\tfrac{1}{2}x) - (3-\tfrac{1}{2}x)^2\, dx = \boxed{18}$$

d. $\nabla \cdot F = e^z + e^z + e^z = 3e^z$

$$\int_0^6\int_0^4\int_0^4 3e^z\, dz\, dy\, dx = \int_0^6\int_0^4 3e^z\Big|_0^4 dy\, dx = 3(e^4-1)\cdot 4 \cdot 6 = \boxed{72(e^4-1)}$$

9. a sink has a net negative flow, a source a net positive flow
an incompressible flow has a net zero flow.
pos/neg/0 determined by flow through closed surface
(can be calculated by Divergence theorem)

10.
cannot be determined from general properties since
$\nabla \cdot F = \nabla \cdot (\nabla f) = \nabla^2 f$ is the laplacian.
this will be zero if potential function is linear.

11. $F = 4x + z^2 - y$ $\quad \nabla F = \langle 4, -1, 2z \rangle$ $\quad \|\nabla F\| = \sqrt{16+1+4z^2}$
$= \sqrt{17+4z^2}$

$$\int_0^1\int_0^1 \sqrt{17+4z^2}\, dz\, dx = \int_0^1 \sqrt{17+4z^2}\, dz \approx 4.2795$$

12. $r_u = 2u\hat{i} + v\hat{j} + 0\hat{k}$
$r_v = 0\hat{i} + u\hat{j} + v\hat{k}$

$$r_u \times r_v = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2u & v & 0 \\ 0 & u & v \end{vmatrix} = (v^2-0)\hat{i} - (2uv-0)\hat{j} + (2u^2-0)\hat{k}$$

$\langle v^2, -2uv, 2u^2 \rangle$

$$\|r_u \times r_v\| = \sqrt{v^4 + 4u^2v^2 + 4u^4} = \sqrt{(v^2+2u^2)^2} = v^2 + 2u^2$$

$$\int_0^2\int_0^1 v^2 + 2u^2\, du\, dv = \int_0^2 v^2u + \tfrac{2}{3}u^3\Big|_0^1 dv = \int_0^2 v^2 + \tfrac{2}{3}\, dv = \tfrac{1}{3}v^3 + \tfrac{2}{3}v\Big|_0^2 =$$

$$\tfrac{8}{3} + \tfrac{4}{3} = \tfrac{12}{3} = \boxed{4}$$

13a.

$r(u,v) = 2\cos u \sin v\,\hat{i} + 2\sin u \sin v\,\hat{j} + 2\cos v\,\hat{k}$ $\qquad 0 \le u \le 2\pi$, $\; 0 \le v \le \pi$

$r_u = -2\sin u \sin v\,\hat{i} + 2\cos u \sin v\,\hat{j} + 0\hat{k}$

$r_v = 2\cos u \cos v\,\hat{i} + 2\sin u \cos v\,\hat{j} - 2\sin v\,\hat{k}$

13a cont'd

$$r_u \times r_v = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ -2\sin u \sin v & 2\cos u \sin v & 0 \\ 2\cos u \cos v & 2\sin u \cos v & -2\sin v \end{vmatrix} =$$

$$(-4\cos u \sin^2 v - 0)\hat{i} - (4\sin u \sin^2 v - 0)\hat{j} + (-4\sin^2 u \sin v \cos v - 4\cos^2 u \sin v \cos v)\hat{k}$$

$$\|r_u \times r_v\| = \sqrt{\underbrace{16\cos^2 u \sin^4 v + 16\sin^2 u \sin^4 v}_{=1} + 16\sin^2 v \cos^2 v} =$$

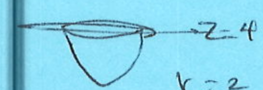
$$= \sqrt{16\sin^4 v + 16\sin^2 v \cos^2 v} = \sqrt{16\sin^2 v(\underbrace{\sin^2 v + \cos^2 v}_{=1})} = 4\sin v$$

$$\int_0^{2\pi}\int_0^2 (\underbrace{4\cos^2 u \sin^2 v \cdot 2\cos v + 4\sin^2 u \sin^2 v}_{=1} \cdot 2\cos v)\, 4\sin v \, du\, dv =$$

$$\int_0^{2\pi}\int_0^2 (8\sin^2 v \cos v) 4\sin v \, du\, dv = 32\int_0^{2\pi} \sin^3 v \cos v \cdot 2\, dv =$$

$$64 \, \frac{1}{4}\sin^4 v \Big|_0^{2\pi} = \boxed{0}$$

b. $\nabla \cdot F = y^2 + 0 + x^2 = x^2 + y^2$



$$\int_0^{2\pi}\int_0^2\int_{x^2+y^2=r^2}^4 r^2 \cdot r\, dz\, dr\, d\theta =$$

$$\int_0^{2\pi}\int_0^2 r^3\Big|_{r^2}^4 dr\, d\theta = \int_0^{2\pi}\int_0^2 4r^3 - r^5\, dr\, d\theta = \int_0^{2\pi} r^4 - \frac{1}{6}r^6\Big|_0^2 d\theta =$$

$$\left(16 - \frac{64}{6}\right)\cdot 2\pi = \boxed{\frac{32\pi}{3}}$$