MTH 291, Exam #1, Spring 2018 Name KEY

Instructions: Show all work. Give exact answers unless specifically asked to round. All complex numbers should be stated in standard form, and all complex fractions should be simplified. If you do not show work, problems will be graded as "all or nothing" for the answer only; partial credit will not be possible and any credit awarded for the work will not be available.

1. Show that every member of the family of functions $y = \frac{\ln x + C}{x}$ is a solution of the differential equation $x^2y' + xy = 1$. (10 points)

$$y' = \frac{\frac{1}{x}\cdot x - (\ln x + C)}{x^2} = \frac{1 + \ln x + C}{x^2}$$

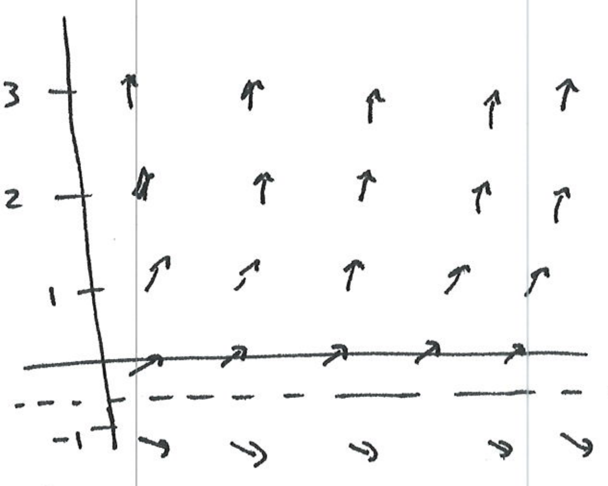
$$\cancel{x^2}\left(\frac{1-(\ln x + C)}{\cancel{x^2}}\right) + \cancel{x}\left(\frac{\ln x + C}{\cancel{x}}\right) = 1$$

$$1 - \cancel{\ln x} - \cancel{C} + \cancel{\ln x} + \cancel{C} = 1 \quad \checkmark$$

2. Graph the direction field for the equation $y' = 1 + 2y$ by hand. Note any equilibria. (10 points)

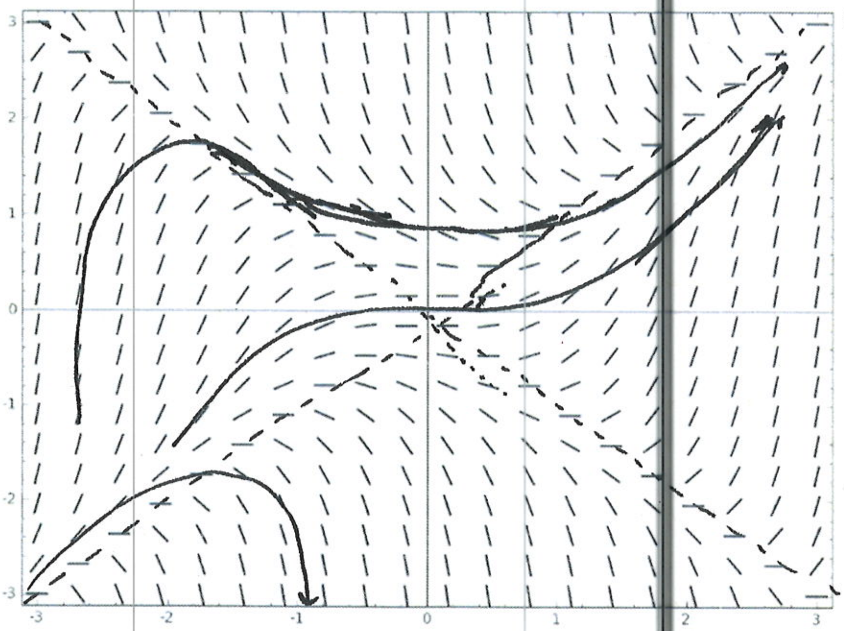
$0 = 1 + 2y$

$y = \frac{-1}{2}$



$y = -1/2$ equilibrium

3. For each slope field shown below, select two different sets of initial conditions and plot the trajectory of the graph from that point. Note any equilibria (points or lines where the slope is zero). (5 points each)

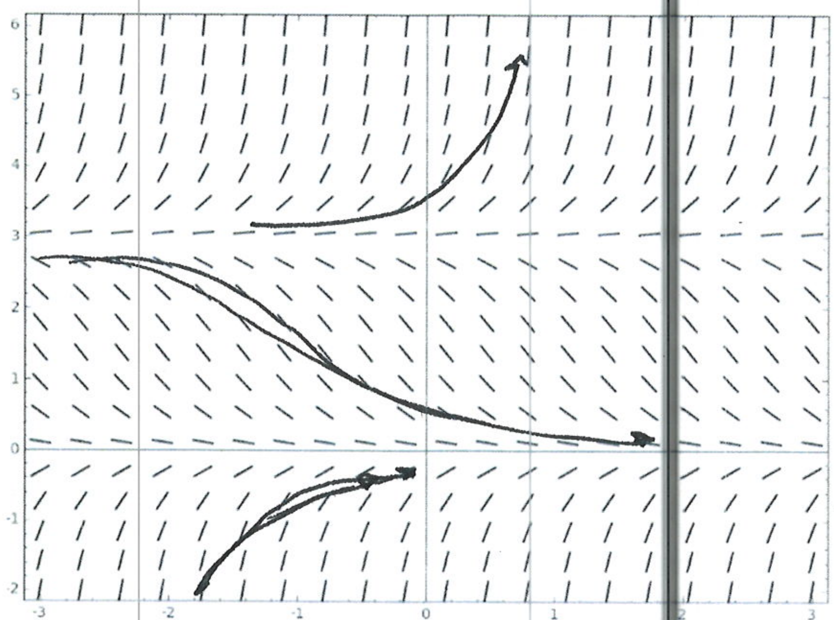


← zero slopes

answers will vary

zero slopes

a.



equilibrium $y=3$ unstable

$y=0$ equilibrium Stable

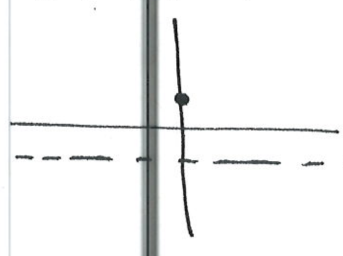
b.

4. Use the properties of the Uniqueness and Existence Theorem to determine where the differential equation $y' = \frac{xy\sin x}{y+1}, y(0) = 1$ is guaranteed to have a solution. Sketch the graph of the region and locate the initial condition on the graph. (12 points)

$f(x,y) = \dfrac{xy\sin x}{y+1}$

$f_y = \dfrac{x\sin x - xy\sin x}{(y+1)^2}$

not defined at $\boxed{y=-1}$



↙ defined for $(y(0))=1$

undefined

5. Use Euler's Method with the indicated step size to find an estimate of the solution at the indicated point: $y' = y - 2x, y(1) = 0, \Delta x = 0.5, y(3) = ?$ (15 points)

n	x_n	y_n	m_n	$y_{n+1} = m_n \Delta x + y_n$
0	1	0	-2	$-2 * .5 + 0 = -1$
1	1.5	-1	$-1 - 2(1.5) = -4$	$-4 * .5 + -1 = -3$
2	2	-3	$-3 - 2(2) = -7$	$-7 * .5 - 3 = -6.5$
3	2.5	-6.5	$-6.5 - 2(2.5) = -11.5$	$-11.5 * .5 - 6.5 = -12.25$
4	3	-12.5		

$$\boxed{y(3) \approx -12.5}$$

6. Solve the differential equation $\frac{dy}{dx} = \frac{\ln x}{xy}, y(1) = 2$ using separation of variables. (12 points)

$$\int y\,dy = \int \frac{\ln x}{x}\,dx$$

$u = \ln x$

$du = \frac{1}{x}dx$

$$\frac{1}{2}y^2 = \frac{(\ln x)^2}{2} + C$$

$$\boxed{y^2 = (\ln x)^2 + C}$$

7. Initially a tank contains 1kg of salt dissolved in 100L of water. Salty water containing 1/4kg/L at a rate of 3L/min is added to the tank and the (stirred) solution is draining from the tank at 3L/min. Determine an equation for how much salt is in the tank at any time t. Sketch the graph of the solution. (15 points)

$A(0) = 1\text{kg}$

$$\frac{dA}{dt} = \frac{1}{4}\frac{\text{kg}}{\text{L}} \cdot \frac{3\text{L}}{\text{min}} - \frac{A\text{ kg}}{100\text{ L}}\frac{3\text{L}}{\text{min}}$$

$$\frac{dA}{dt} = 3/4 - \frac{3A}{100} = -\frac{3}{100}(A - 25)$$

$$\int \frac{dA}{A-25} = \int -\frac{3}{100}\,dt$$

$$\ln|A - 25| = -3/100\,t + C$$

$$A - 25 = e^{-3/100\,t} A_0$$

$$A(t) = 25 + A_0 e^{-3/100\,t}$$

$$\boxed{A(t) = 25 - 24e^{-3/100\,t}}$$

8. Solve the linear differential equation $y' + 4y = 2e^{x}$ by the method of integrating factors (reverse product rule). (12 points)

$\mu = e^{\int 4\,dx} \cdot e^{4x}$

$e^{4x} y' + 4e^{4x} y = 2e^{x} \cdot e^{4x}$

$\int (e^{4x} y)' = \int 2e^{5x}$

$e^{4x} y = \frac{2}{5} e^{5x} + C$

$\boxed{y = \frac{2}{5} e^{x} + Ce^{-4x}}$

9. Solve the homogeneous equation $(x^3 + y^3)dx - xy^2 dy = 0$. Stop when the equation is separable. (10 points)

$(x^3 + y^3)dx = xy^2 dy$

$y = vx \rightarrow v = y/x$

$y' = v'x + v$

$\frac{dy}{dx} = \frac{x^3 + y^3}{xy^2}$

$v'x + v = \frac{x^3 + v^3x^3}{x v^2 x^2} = \frac{\cancel{x^3}(1+v^3)}{\cancel{x^3}\, v^2}$

$v'x + \cancel{v} = \frac{1+v^3}{v^2} - v \cdot \frac{v^2}{v^2}$

$v'x = \frac{1 + \cancel{v^3} - \cancel{v^3}}{v^2} \Rightarrow v'x = \frac{1}{v^2}$

$\int v^2\,dv = \int \frac{1}{x}\,dx$

$\frac{1}{3} v^3 = \ln x + C$

$\frac{1}{3}\left(\frac{y^3}{x^3}\right) = \ln x + C$

$\boxed{y^3 = 3x^3 \ln x + Cx^3}$

10. Solve the Bernoulli equation $y' + \frac{1}{x}y = x\sqrt{y}$. Stop when the equation is linear. (10 points)

$y^n = y^{1/2}$

$(1-\frac{1}{2})y^{-1/2} = \frac{1}{2}y^{-1/2}$

$z = y^{1/2}$

$z' = \frac{dz}{dt} = \frac{1}{2}y^{-1/2}y'$

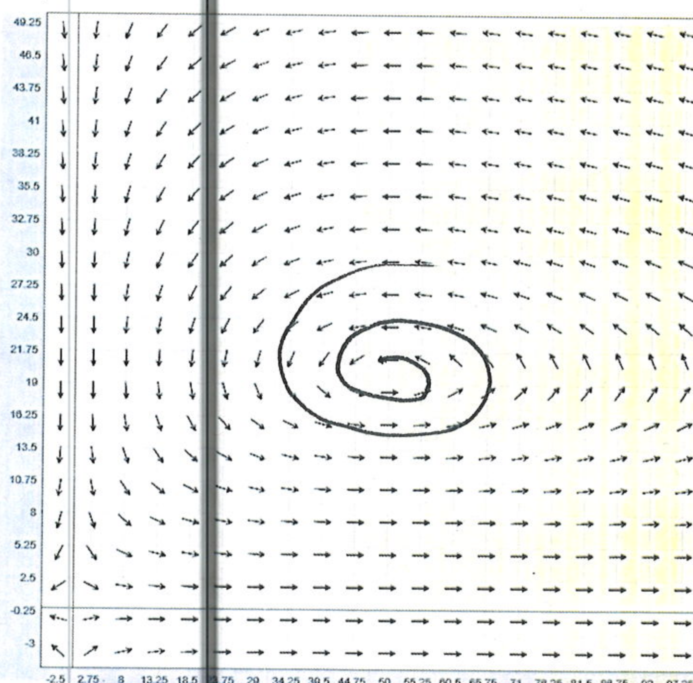
$\frac{1}{2}y^{-1/2}y' + \frac{1}{x}\left(\frac{1}{2}y^{1/2}\right) = x \cdot \frac{1}{2}$

$\boxed{z' + \frac{1}{2x}z = \frac{1}{2}x}$

linear

11. Consider the system of differential equations $\begin{cases} \frac{dx}{dt} = 0.8x - 0.04xy \\ \frac{dy}{dt} = -0.3y + 0.006xy \end{cases}$, $x(0) = 55, y(0) = 10$.

The slope field for the system is shown below. Describe the behavior of the system. Identify and equilibria and characterize its stability. Describe the behavior of each population in the absence of the other one. (16 points)



Spirals in toward equilibrium (stable)

at approximately (50, 20)

X grows alone

Y dies alone

System appears to model a predator-prey relationship