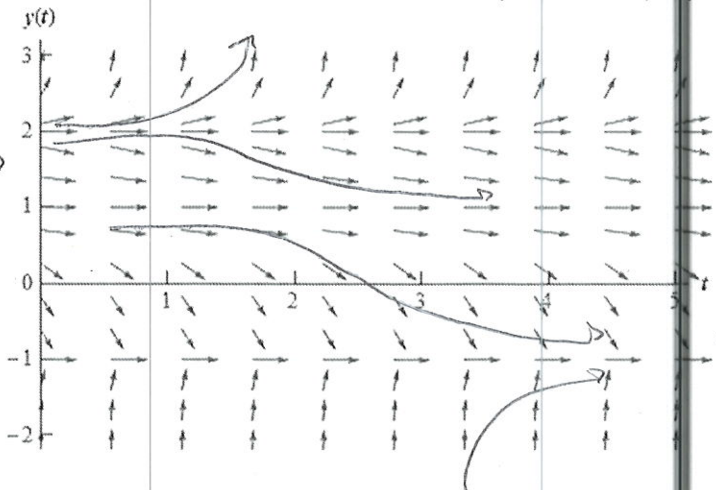
MTH 291, Exam #2, Spring 2018 Name KEY

Instructions: Show all work. Give exact answers unless specifically asked to round. All complex numbers should be stated in standard form, and all complex fractions should be simplified. If you do not show work, problems will be graded as "all or nothing" for the answer only; partial credit will not be possible and any credit awarded for the work will not be available.

1. Direction fields for population models are shown below. Find a differential equation that models the population (up to a constant multiple). Plot trajectories of initial conditions that models each type of trajectory for the model. Where is the model logistic? Describe the long-term behavior of each trajectory. Describe each equilibrium as a carrying capacity or a threshold. Convert each direction field into a phase line. (12 points each)



logistic

a.

increases to ∞

y=2 threshold

goes to 1 $\frac{dy}{dt} = (y-2)(y+1)(y-1)^2$

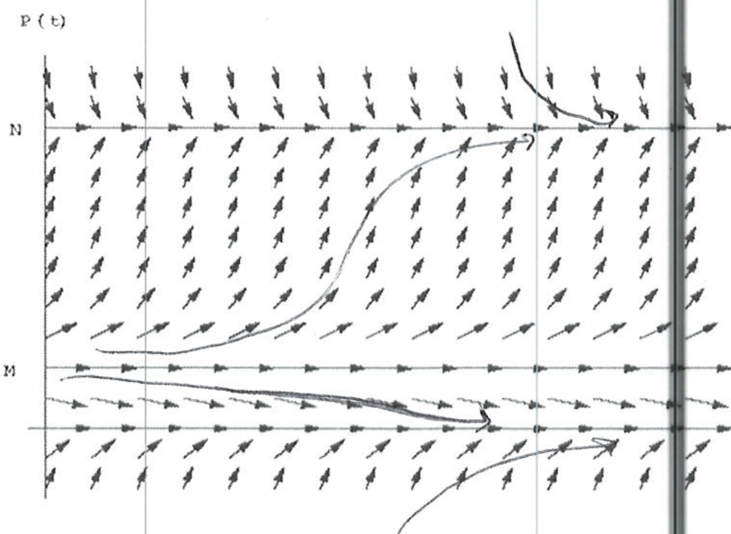
Y=1

neither

goes to -1

Y=-1 neither

goes to -1



b.

goes to N

Y=N carrying capacity

goes to N

logistic

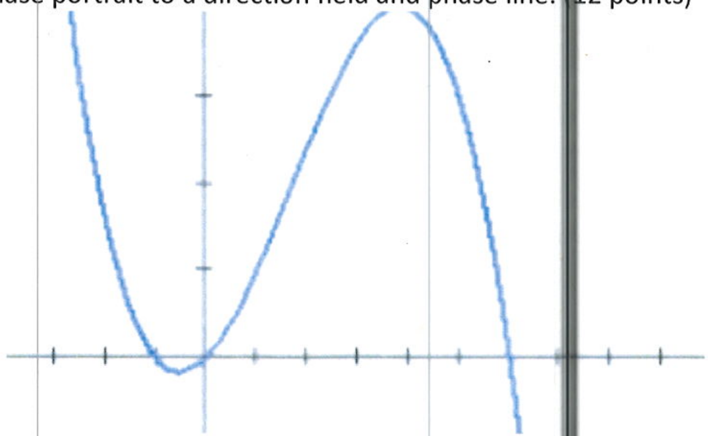
Y=M threshold

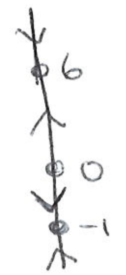
goes to 0 logistic

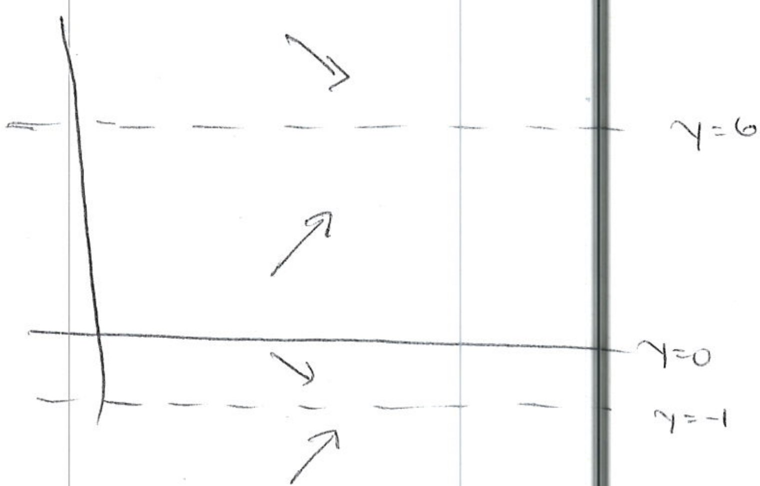
Y=0

goes to 0

2. The phase portrait for the differential equation $y' = -y(y+1)(y-6)$ is shown below. Convert the phase portrait to a direction field and phase line. (12 points)







3. A force of 400 N stretches a spring 2 meters. A mass of 50 kg is attached to the end of the spring and is initially released from equilibrium position with an upward velocity of 10 m/s. Write the second-order equation that models the system. (7 points)

$F = kx \quad 400 = k(2) \quad M = 50 \quad y(0) = 0 \quad y'(0) = 10$

$k = 200$

$50y'' + 200y = 0$

a. A dashpot device is added to the system to provide damping numerically equivalent to twice the velocity. Write the new second-order system. (5 points)

$\gamma = 2$

$50y'' + 2y' + 200y = 0$

b. Convert the second-order equation in (a) to a first order system. Solve the system. (12 points)

$$25y'' + y' + 100y = 0$$

$$25y'' = -100y - y'$$

$$y'' = -4y - \frac{1}{25}y'$$

$$y = x_1$$

$$x_1' = y' = x_2$$

$$x_2' = y''$$

$$\boxed{y = 5e^{-.02t}\sin(2t)}$$

$$25r^2 + r + 100 = 0$$

$$r = \frac{-1 \pm \sqrt{1 - 4(25)(100)}}{50}$$

$$= \frac{-1 \pm \sqrt{9999}\,i}{50} = -.02 \pm 1.9999i$$

$$x_1' = x_2$$

$$x_2' = -4x_1 - \frac{1}{25}x_2$$

$y'(0) = 10$

$y(0) = 0$

$$y = c_1 e^{-.02t}\cos(1.9999t) + c_2 e^{-.02t}\sin(1.9999t)$$

$$y' = -.02c_2 e^{-.02t}\sin(1.9999t) + c_2 e^{-.02t}\cos(1.9999t)(1.9999) \qquad \frac{10}{1.9999} = c_2$$

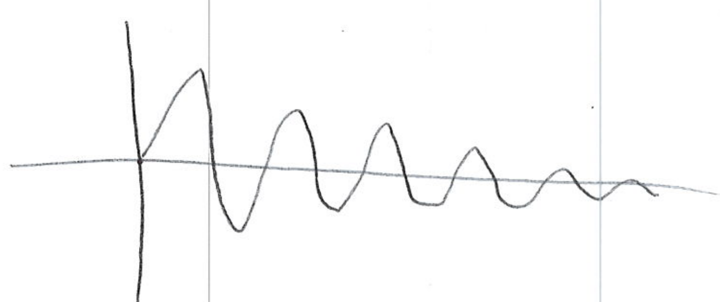
c. Is the damping underdamped, critically damped or overdamped? Explain your reasoning. (6 points)

underdamped

exponentially decaying oscillation

Solution complex but not pure imaginary

d. Sketch the graph of the solution for the given initial conditions. (6 points)



4. Consider the competition model $\begin{cases} \frac{dx}{dt} = 0.8x + 0.4x^2 - xy \\ \frac{dy}{dt} = 0.3y - 0.6y^2 - xy \end{cases}$, $x(0) = 7, y(0) = 6$. Sketch the nullclines for the system and use them to identify any equilibria. Using the information obtained from the nullclines, can you characterize the equilibria as stable, unstable or a saddle point? (20 points)

$$\frac{dx}{dt} = x(.8 + .4x - y)$$

$x = 0$

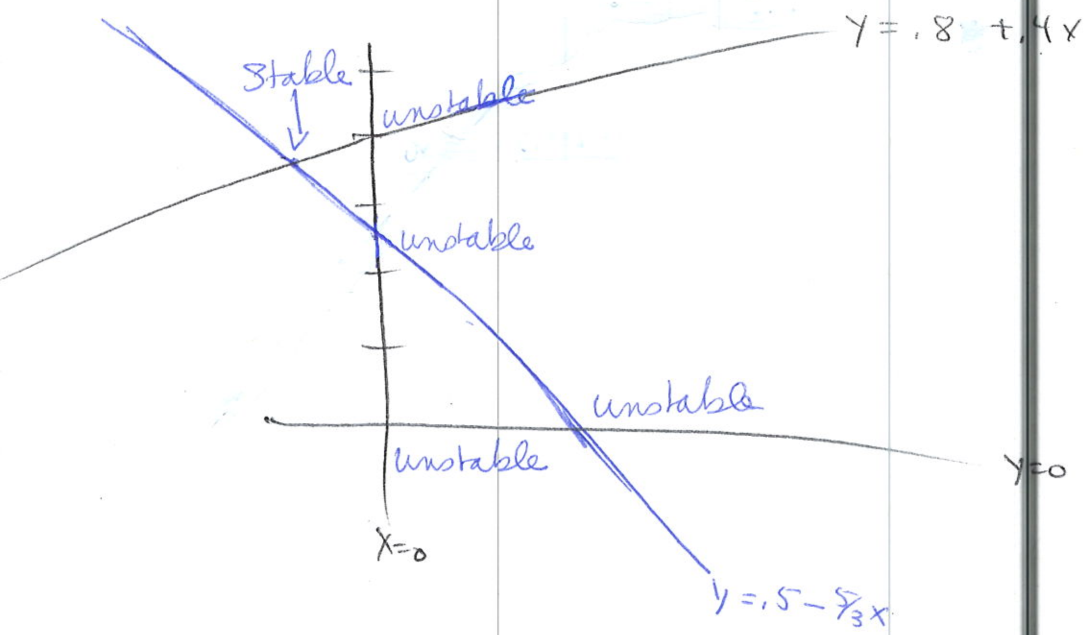
$y = .8 + .4x$

$$\frac{dy}{dt} = y(.3 - .6y - x)$$

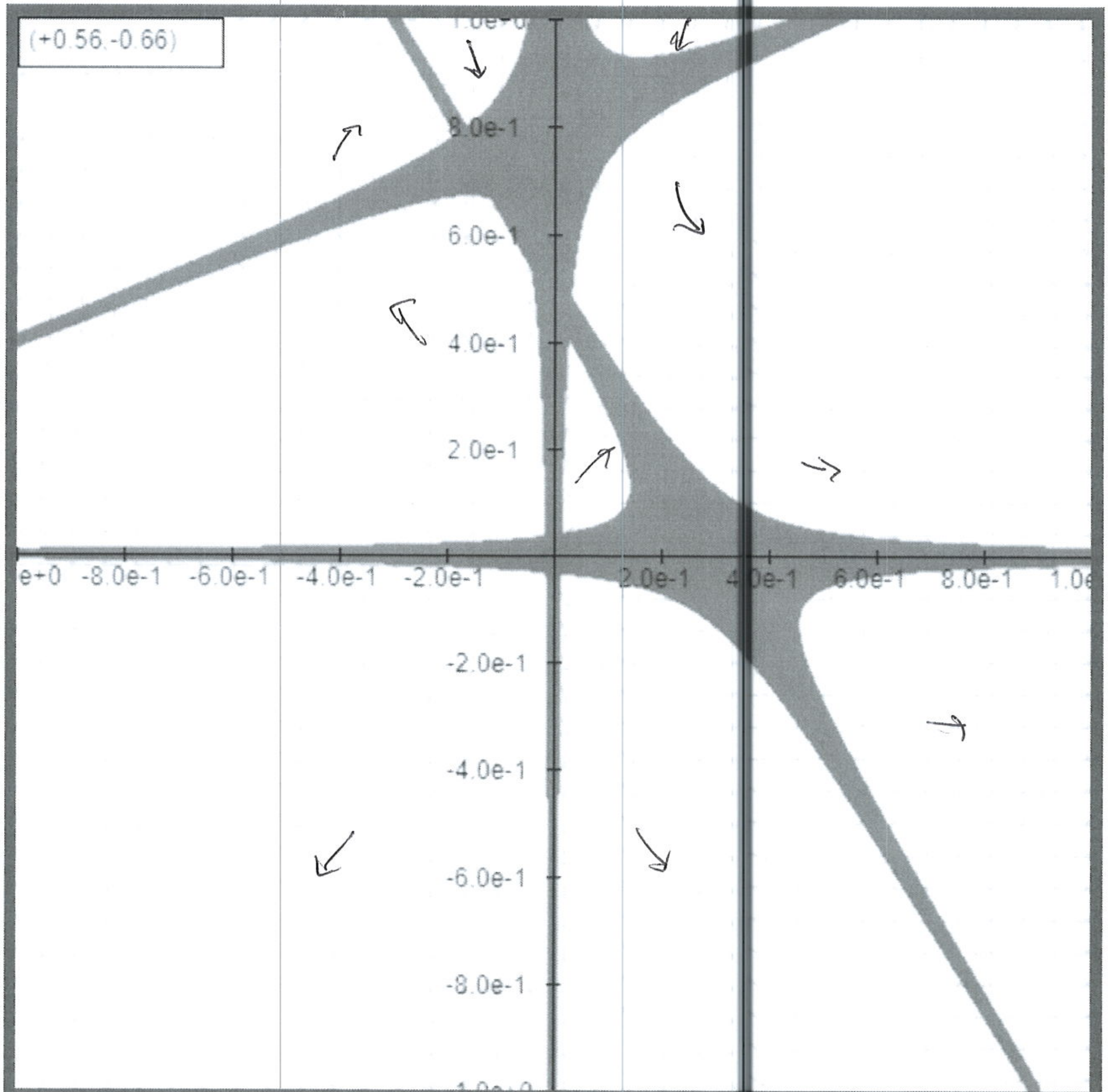
$y = 0$

$$\frac{.6y}{.6} = \frac{.3 - x}{.6}$$

$$y = .5 - \frac{5}{3}x$$



intersections marked "unstable" may be saddle points



(+0.56,-0.66)
1.0e+0
8.0e-1
6.0e-1
4.0e-1
2.0e-1
-2.0e-1
-4.0e-1
-6.0e-1
-8.0e-1
-1.0e+0
-8.0e-1 -6.0e-1 -4.0e-1 -2.0e-1 2.0e-1 4.0e-1 6.0e-1 8.0e-1

5. Solve the systems of equations for the general solution below using eigenvalues. Be sure that your solutions are expressed only with real-valued functions. (14 points each)

a. $\vec{x}'(t) = \begin{pmatrix} 6 & -1 \\ 5 & 4 \end{pmatrix} \vec{x}$

$(6-\lambda)(4-\lambda)+5=0$

$\lambda^2 - 10\lambda + 24 + 5 = 0$

$\lambda^2 - 10\lambda + 29 = 0$

$\lambda = \dfrac{10 \pm \sqrt{100-4(29)}}{2} \leftarrow \sqrt{-16}$ =

$\dfrac{10 \pm 4i}{2} = 5 \pm 2i$

$\begin{bmatrix} 6-(5+2i) & -1 \\ 5 & 4-(5+2i) \end{bmatrix} = \begin{bmatrix} 1-2i & -1 \\ 5 & -1-2i \end{bmatrix}$

$x_1 = \dfrac{(1+2i)x_2}{5}$

$\vec{v} = \begin{bmatrix} 1+2i \\ 5 \end{bmatrix}$

$e^{5t}\begin{bmatrix} 1+2i \\ 5 \end{bmatrix}(\cos 2t + i \sin 2t) =$

$e^{5t}\begin{bmatrix} \cos 2t + i\sin 2t + 2i\cos 2t - 2\sin 2t \\ 5\cos 2t + i5\sin 2t \end{bmatrix}$

$\vec{X} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = c_1 e^{5t}\begin{pmatrix} \cos 2t - 2\sin 2t \\ 5\cos 2t \end{pmatrix} + c_2 e^{5t}\begin{pmatrix} \sin 2t + 2\cos 2t \\ 5\sin 2t \end{pmatrix}$

b. $\vec{x}'(t) = \begin{pmatrix} -6 & 2 \\ -3 & 1 \end{pmatrix} \vec{x}$

$(-6-\lambda)(1-\lambda)+6=0$

$\lambda^2 + 5\lambda - 6 + 6 = 0$

$\lambda^2 + 5\lambda = 0$

$\lambda = 0, \lambda = -5$

$\lambda = 0 \quad \begin{bmatrix} -6 & 2 \\ -3 & 1 \end{bmatrix}$

$-3x_1 = -x_2$

$x_1 = \frac{1}{3}x_2 \qquad \vec{v}_1 = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$

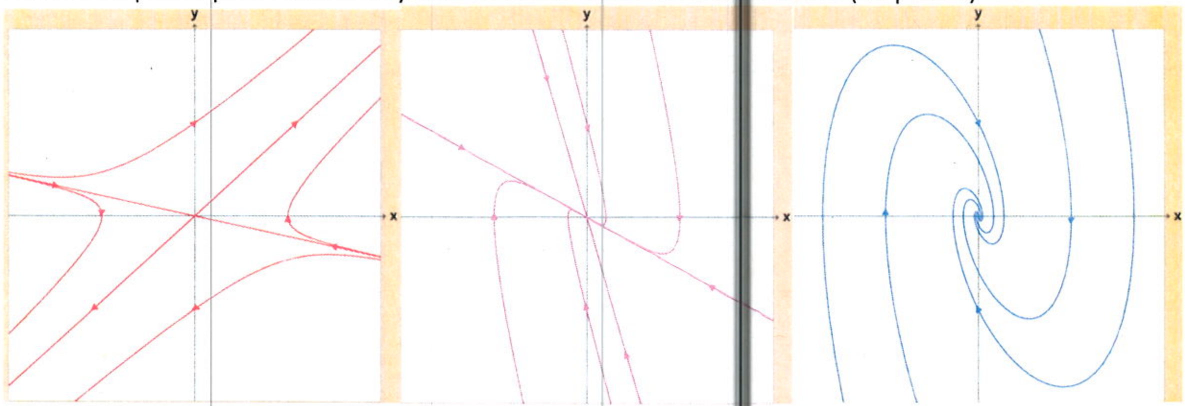
$\lambda = -5 \quad \begin{bmatrix} -6+5 & 2 \\ -3 & 1+5 \end{bmatrix} = \begin{bmatrix} -1 & 2 \\ -3 & 6 \end{bmatrix}$

$-x_1 = -2x_2$

$x_1 = 2x_2 \qquad \vec{v}_2 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$

$\vec{X} = c_1\begin{bmatrix} 1 \\ 3 \end{bmatrix} + c_2\begin{bmatrix} 2 \\ 1 \end{bmatrix}e^{-5t}$

6. A series of phase portraits for a system of linear ODEs is shown below. (12 points)



A B C

One of these phase portraits is not a possible solution for a system of springs. Which one? Explain your reasoning.

A is not possible since the origin is a repeller. This would represent exponential growth, but all spring problems are pure oscillators (no damping) or exponentially decaying.

exponential growth would be negative damping or adding energy to the system.