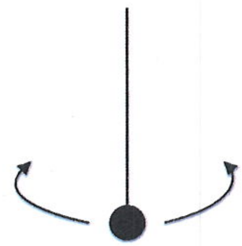
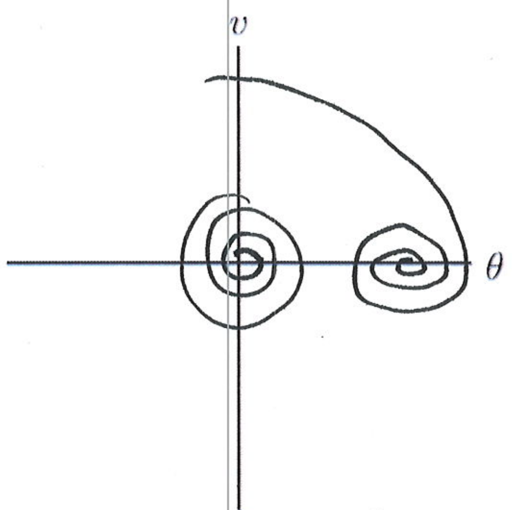
In the Swing of Things

A pendulum is attached to a wall in such a way that it is free to rotate around in a complete circle. Without provocation, Debra takes a baseball bat and hits it, giving it an initial velocity and setting it in motion.



$\theta = 0$

1. If we call θ the angular position of the pendulum (where $\theta = 0$ corresponds to when the pendulum is hanging straight down) and we call the velocity of the pendulum v, what would angular position versus velocity graphs look like for a variety of different initial velocities due to Debra's hit? Provide a brief description of the motion of the pendulum for your graphs.



depending on strength of hit

2. How many equilibrium solutions are there, where are they, and how would you classify them?

stable equilibria at 0 ± n(rotations)

unstable equilibria at ± ½ rotations

Linearization and Linear Stability Analysis

In the next several questions we will develop tools to analyze equilibria of nonlinear systems. To do this, we will first build our intuition by studying first order nonlinear equations.

7. Recall from Calculus that the linearization, $L(h)$, of a function around a point of interest, x^*, is given by $L(h) \equiv f(x^*) + hf'(x^*)$. The key feature of the linearization is that, when $x \approx x^*$, that is, $x = x^* + h$ for $h \approx 0$, then $f(x) \approx L(h)$.

Find the linearization of $f(x) = 1 - x^2$ around $x^* = 1$. $f(1) = 1 - 1^2 = 0$	$f'(x) = -2x$ $f'(x^*) = (x^*+h)(-2)$ $= -2x^* \cancel{-2h}^{\to 0}$ $f'(1) = -2$ $hf'(x^*) = -2h + 0 = -2h$
If $x \approx 1$, x can be written as $x = 1 + h$ where $h \approx 0$. Suppose x follows the differential equation $\frac{dx}{dt} = 1 - x^2$. Use the linearization above to write down a linear differential equation for $\frac{dh}{dt}$.	$1 - (1+h)^2 =$ $1 - (1 + 2h + h^2) =$ $\cancel{1} \cancel{-1} - 2h - h^2 = \underline{\underline{-2h}} - h^2$ linear $\frac{dx}{dt} = \frac{dh}{dt}$ $-2h$ $\quad \frac{dh}{dt} = -2h$
According to the above differential equation, what is the long term behavior of h?	$\frac{dh}{h} = -2\,dt \to h = e^{-2t} \to 0$
If $x(0) \approx 1$, what does the long term behavior of h tell you about the long term behavior of x?	nearby, they will behave like the linearized function

8. (a) Consider again $\dfrac{dx}{dt} = 1 - x^2$, but this time with $x(0) \approx -1$. Find a new linearization and use it to make a long term prediction about x.

$f'(x) = -2x \quad x = -1 \to f'(-1) = 2$

$\frac{dh}{dt} = 2h \to h = e^{2t} \to \infty$ unstable

Applying Newton's 2nd Law of motion (where $\theta = 0$ corresponds to the downward vertical position and counterclockwise corresponds to positive angles θ) yields the differential equation

$$\frac{d^2\theta}{dt^2} + \frac{b}{m}\frac{d\theta}{dt} + \frac{g}{l}\sin(\theta) = 0$$

where b is the coefficient of damping, m is the mass of the pendulum, g is the gravity constant, and l is the length of the pendulum (See homework problem 5 for a derivation of this equation). Estimating the parameter values for the pendulum that Debra hits and changing this second order differential equation to a system of differential equations yields

$$\begin{aligned}\frac{d\theta}{dt} &= v\\ \frac{dv}{dt} &= -0.2v - \sin(\theta)\end{aligned}$$

3. How many equilibrium solutions does this system of differential equations have, where are they, and based on the context what types of equilibrium solutions would you expect them to be? How does this connect with your answer to 2?

the sine term is what produces the extra equilbria

$v=0$ and $\sin\theta = 0$ $\quad n\frac{\pi}{2} = \theta$

4. You might recall that if θ is small, $\sin(\theta) \approx \theta$. Explain why this is true and then use this fact to approximate the above system with a linear system and classify the equilibrium solution at the origin.

taylor series for $\sin x = x + \frac{x^3}{3} + \ldots$

when θ is small $\frac{\theta^3}{3}$ is much! smaller (for $\theta < 1$)

5. Classify the equilibrium point at $\theta = \pi$.

unstable  corresponds to standing straight up

6. Use the GeoGebra applet, https://ggbm.at/SpfDSc5Q, to approximate the range of initial velocities with zero initial displacement that will result in the pendulum making exactly one complete rotation before eventually coming to rest.

answers may vary somewhat

positive and negative velocities are possible

(b) Why was it necessary to construct a **new** linearization to study $x(0) \approx -1$?

because the old linearization
was based on a different center

(c) Using linearization to determine the stability of a critical point is called "linear stability analysis." Use a phase line to corroborate your linear stability analysis.

(d) For an arbitrary system, $\dfrac{dx}{dt} = f(x)$ with an equilibrium point at $x = x^*$, describe how you can use linear stability analysis to determine the stability of the equilibrium point.

find derivative
evaluate at critical point and linearize
replace and solve

9. Consider the following system:

$$\frac{dx}{dt} = 1 - x^2 \quad \text{nonlinear}$$
$$\frac{dy}{dt} = -3x - 3y \quad \text{linear}$$

(a) Algebraically find the equilibrium solutions.

$1 - x^2 = 0$ $\quad x = \pm 1$ $\quad x = 1$ $\quad x = -1$

$0 = -3x - 3y$ $\quad y = -x$ $\quad y = -1$ $\quad y = 1$

(b) Tanesha used the GeoGebra Vector field applet, https://ggbm.at/kkNXUVds, to plot the vector field associated with the differential equation. Based on this vector field, how would you classify the equilibria?

$(1, -1)$
Stable

$(-1, 1)$
unstable or saddle

We can also perform linear stability analysis on a system of two or more variables, such as the one in the previous problem. Consider a function $f(x,y)$, then Taylor's theorem states that, if $(x,y) \approx (x^*,y^*)$, that is, if $(x,y) = (x^* + h_1, y^* + h_2)$ where $h_1 \approx 0$ and $h_2 \approx 0$, then

$$f(x,y) \approx L(h_1,h_2) = f(x^*,y^*) + h_1 f_x(x^*,y^*) + h_2 f_y(x^*,y^*)$$

where f_x and f_y are the partial derivatives of f with respect to x and y, respectively.

$L(h_1,h_2)$ is called the linearization of $f(x,y)$ around (x^*,y^*).

10. Consider the system of differential equations:

$$\frac{dx}{dt} = 1 - x^2$$
$$\frac{dy}{dt} = -3x - 3y$$

Let's first study the equilibrium at (1,-1).

If the system had $x(0) \approx 1$ and $y(0) \approx -1$, we could write $x = 1 + h_1$ and $y = -1 + h_2$, with $h_1 \approx 0$ and $h_2 \approx 0$. Use the linearization of the original system of equations around $(1,-1)$ to write down a system of differential equations for h_1 and h_2	$f(x) = 1 - x^2$ $f'(x) = -2x$ at $(1,-1) \rightarrow -2$ $\frac{dh}{dt} = -2h_1$ $\frac{dy}{dt} = -3h_1 - 3y$
What are the long term behaviors of h_1 and h_2?	y is linear $\rightarrow 0$ $x \rightarrow 0$
What can you conclude about the long term behaviors of x and y?	$\rightarrow$ critical point
Classify the equilibrium point $(1,-1)$, according to your linear stability analysis.	Stable

$$\begin{vmatrix} -2-\lambda & 0 \\ -3 & -3-\lambda \end{vmatrix} = (-2-\lambda)(-3-\lambda) - 0 = 0$$
$$\lambda = -2, -3$$

11. (a) Consider again

$$\frac{dx}{dt} = 1 - x^2$$
$$\frac{dy}{dt} = -3x - 3y$$

Use linear stability analysis to classify the equilibrium point at $(-1, 1)$.

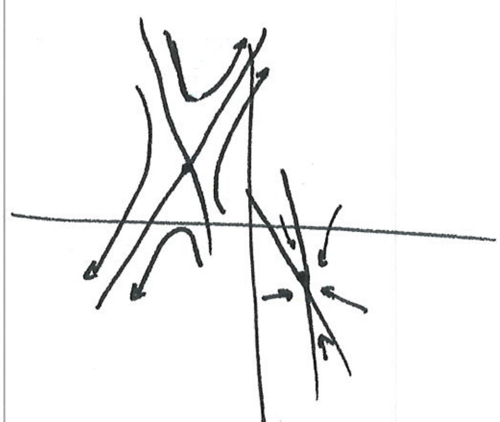
$$\frac{dh}{dt} = 2h$$
$$\frac{dy}{dt} = -3h - 3y$$

$$\begin{vmatrix} 2-\lambda & 0 \\ -3 & -3-\lambda \end{vmatrix}$$

$$(2-\lambda)(-3-\lambda) = 0$$
$$\lambda = 2, \lambda = -3$$

Saddle point

(b) Combine your results from question 10 and 11a, to sketch a possible phase plane for the system of differential equations. Does an analysis of the system using nullclines corroborate your linear stability analysis?



12. For a system of differential equations

$$\frac{dx}{dt} = f(x,y)$$
$$\frac{dy}{dt} = g(x,y)$$

with an equilibrium point at (x^*, y^*), the matrix

$$J = \begin{bmatrix} f_x(x^*,y^*) & f_y(x^*,y^*) \\ g_x(x^*,y^*) & g_y(x^*,y^*) \end{bmatrix}$$

is called the **Jacobian matrix**. Explain how you can use the Jacobian matrix to determine the behavior of a the system of differential equations near (x^*, y^*).

the Jacobian matrix is like the linearization of the field near the critical points

find eigenvalues to determine behavior

13. Use linear stability analysis to classify the critical points you found in the pendulum system.

$$\frac{d\theta}{dt} = v$$
$$\frac{dv}{dt} = -0.2v - \sin(\theta)$$

$$\begin{vmatrix} 0 & 1 \\ -\cos\theta & -.2 \end{vmatrix}$$

$\theta = 0$ $\quad (-\lambda)(-.2-\lambda) + 1 = 0$

$\lambda^2 + .2\lambda + 1 = 0$

$\lambda = \dfrac{-.2 \pm \sqrt{.04 - 4}}{2}$

real negative, complex

Stable

Same for $2n\pi$

$\theta = \pi(2n+1)$

$$\begin{vmatrix} 0 & 1 \\ -\cos\theta & -.2 \end{vmatrix}$$

$(-\lambda)(-.2-\lambda) + (-1) = 0$

$\lambda^2 + .2\lambda - 1 = 0$

$\lambda = \dfrac{-.2 \pm \sqrt{.04+4}}{2}$ real one pos one neg

saddle