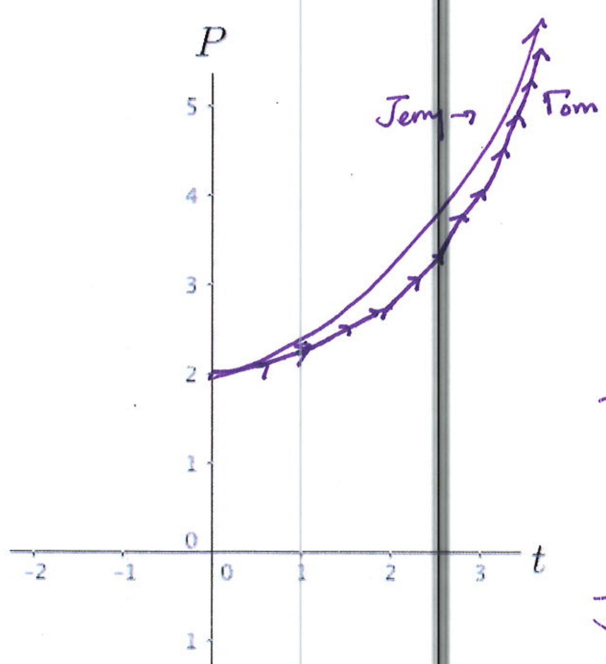
Comparing Predictions

Jerry and Tom are using the differential equation $\frac{dP}{dt} = 0.2P$ to make predictions about the number of a particular species of fish in Lake Michigan. They know that the initial population P is 2 at time $t = 0$ (as before, think of 2 as scaled for say, 2,000 or 20,000).

Although Jerry and Tom have the same goal (to obtain predictions for future fish population), they have different approaches to achieve this goal.

- Tom's approach is to create a graph of the number of fish versus time by connecting slope vectors tip-to-tail, where the rate of change is constant over some time interval, for example $\Delta t = 0.5$.
- Jerry's approach is to create a graph of the number of fish versus time by using a continuously changing rate of change.

1. Sketch Tom and Jerry's approaches below. Will these two approaches result in the same predictions for the number of fish in, say, 2.5 years? If yes, why? If not, how and why will the graphs of their approaches be different?



Tom's is an approximation

Jerry's is exact

in this case, it looks like Tom's underestimates the graph a bit and it may get worse over time.

Separation of Variables

2. **Finding the exact solution.** Jerry's approach involves using a continuously changing rate of change, which corresponds to finding an "exact solution."

   (a) Why do you think the phrase "exact solution" is used to describe the result of Jerry's approach? Explain why it is appropriate to describe the result of Tom's approach as an "approximate solution".

   because the rate of change is tracked and changed "continuously" it adjusts as the equation adjusts

   Tom's approach assumes the slope is approximately the same over some interval, and so can only estimate the curve.

   (b) Use the chain rule to write down, symbolically, the derivative with respect to t of $\ln(P)$, where P is shorthand for $P(t)$.

   $$(\ln P)' = \frac{1}{P} P' = \frac{1}{P} \cdot \frac{dP}{dt}$$

Next you will learn a technique for finding the exact solution corresponding to Jerry's approach. We begin by considering the chain rule.

(c) The following is a method to find the analytic solution to $\frac{dP}{dt} = 0.2P$. For now assume that $P > 0$. This assumption corresponds to the population growth context and it will make the algebra easier and hence the underlying idea clearer.

Divide both sides of $\frac{dP}{dt} = 0.2P$ by P	$\frac{1}{P}\frac{dP}{dt} = 0.2$
Replace $\frac{1}{P}\frac{dP}{dt}$ with $[\ln(P)]'$	$(\ln P)' = 0.2$
Write integrals with respect to t on both sides	$\int (\ln P)' \, dt = \int 0.2 \, dt$
Apply the Fundamental Theorem of Calculus to integrate both sides	$\ln P = 0.2t + C$
Solve for P (and remember that P is actually a function, $P(t)$)	$P = e^{0.2t+C} =$
Show that P can be written as $P(t) = ke^{0.2t}$	$P = e^{0.2t} \cdot e^{C}$ let $e^C = k$ $P(t) = ke^{0.2t}$

$a^b a^c = a^{b+c}$

The end result, $P(t) = ke^{0.2t}$ is called the **general solution** because it represents all possible functions that satisfy the differential equation. We can use the general solution to find any **particular solution**, which is a solution that corresponds to a given initial condition.

3. Use the same technique to find the general solution to $\frac{dy}{dt} = \frac{t}{3y^2}$. The first step is done for you.

$3y^2 \frac{dy}{dt} = t$ $\qquad$ $3y^2 \frac{dy}{dt} = (y^3)'$

$\int (y^3)' \, dt = \int t \, dt \rightarrow y^3 = \frac{1}{2}t^2 + C$

4. In practice, we often circumvent explicit use of the chain rule and instead use a shortcut to more efficiently find the general solution. The shortcut involves treating the derivative $\frac{dP}{dt}$ as a ratio and "separating" the dP and dt. In the table below, follow the instructions to see how the shortcut works, using again the equation $\frac{dP}{dt} = 0.2P$. (See http://kevinboone.net/separation_variables.html) for a nice explanation of the shortcut).

'Separate' the dP from the dt so that dP and P are on the same side. (If there are t's in the equation they must go on the same side as dt.)	$\frac{dP}{P} = 0.2\,dt$
Integrate both sides of the equation (one side with respect to P, the other with respect to t)	$\int \frac{dP}{P} = \ln P$ $\quad \int 0.2\,dt = 0.2t + C$
Continue as before to arrive at a solution of the form $P(t) =$ _____	$\ln P = 0.2t + C$ $P = e^{0.2t+C}$ $P = ke^{0.2t}$ $P(t) = ke^{0.2t}$

5. Use the shortcut to find the general solution to $\frac{dy}{dt} = \frac{t}{3y^2}$.

$$\int 3y^2\,dy = \int t\,dt$$

$$y^3 = \frac{1}{2}t^2 + C$$

Making Connections

7. For the first slope field for $\frac{dL}{dt} = 0.5(1 - L)$ on the following page,
   (a) Using Jerry's approach, sketch as accurately as possible a graph of the solution with initial condition $L(0) = 1/3$.
   (b) Make a copy of this sketch on a transparency.
   (c) If you wanted to obtain the graph of the solution with initial condition $L(0) = 1/2$, how, if at all, might you move the copy of your graph with initial value 1/3 so that it is now a graph of the solution with initial value 1/2? What feature of the differential equation justifies your approach? Shift graph to the left
   (d) Find the general solution for $\dfrac{dL}{dt} = 0.5(1 - L)$ and explain how your results from part 7c can be understood from the general solution. function of L only

$$\frac{dL}{dt} = -0.5(L-1)$$

$$\int \frac{dL}{L-1} = \int -0.5\,dt$$

$$\ln|L-1| = -0.5t + C$$

$$L - 1 = ke^{-0.5t}$$

$$L = 1 + ke^{-0.5t}$$

The graph increases (or decreases) up to $L=1$

new initial condition c

$L = 1 + ke^{-0.5(t-c)}$ shift horizontal

8. For the second slope field for $\frac{dh}{dt} = -t + 1$ on the following page,
   (a) Using Jerry's approach, sketch as accurately as possible a graph of the solution with initial condition $h(0) = 1/2$.
   (b) Make a copy of this sketch on a transparency.
   (c) If you wanted to obtain the graph of the solution with initial condition $h(0) = 1$, how, if at all, might you move the copy of your graph with initial value 1/2 so that it is now a graph of the solution with initial value 1? Explain your idea and provide reasons for why your idea makes sense. Shift up
   (d) Find the general solution for $\frac{dh}{dt} = -t + 1$ and explain how your results from part 8c can be understood from the general solution.

$$\int dh = \int (-t+1)\,dt$$

$h = -\frac{1}{2}t^2 + t + C$ shift vertical

function of t only

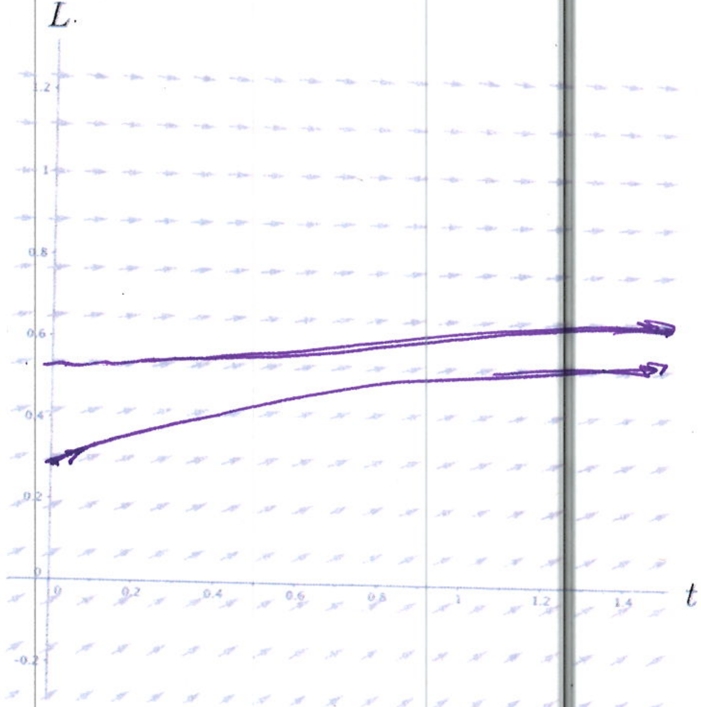
9. Give an example of a differential equation where neither of your ideas from 7c and 8c will work and provide reasons for your response.

$$\frac{dy}{dt} = \frac{t}{t+y}$$

function of y and t both

not separable

Slope Field for $\frac{dL}{dt} = 0.5(1 - L)$



Slope Field for $\frac{dh}{dt} = -t + 1$