A Salty Tank

$S(0)=6$

1. A very large tank initially contains 15 gallons of saltwater containing 6 pounds of salt. Saltwater containing 1 pound of salt per gallon is pumped into the top of the tank at a rate of 2 gallons per minute, while a well-mixed solution leaves the bottom of the tank at a rate of 1 gallon per minute.

   (a) Should the rate of change equation for this situation depend just on the amount of salt S in the tank, the time t, or both S and t? Explain your reasoning.

   it will depend on both S and t

   Since less water is coming out of the tank than going in the amount of water in the tank is changing in time

   (b) The following is a general rule of thumb for setting up rate of change equations for situations like this where there is an input and an output:

   $$\text{rate of change} = \text{rate of change in} - \text{rate of change out}$$

   Using the above rule of thumb, figure out a rate of change equation for this situation.
   Hint: Think about what the *units* of $\frac{dS}{dt}$ need to be, where S is the amount of salt in the tank in pounds.

   lbs   lbs $S(0)=6$

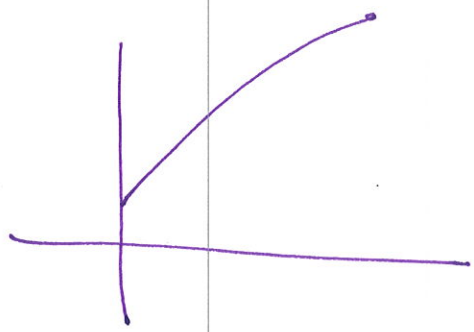
   $$\frac{dS}{dt} = \frac{1\text{ lbs}}{\text{gal}}\cdot\frac{2\text{ gal}}{\text{min}} - \frac{S}{15+t}\cdot\frac{1\text{ gal}}{\text{min}}$$

   min   gal

   $$\frac{dS}{dt} = 2 - \frac{S}{15+t}$$

(c) Use the slope field for this differential equation in the GeoGebra applet, https://ggbm.at/PFRcbkbZ, to sketch a graph of the solution with initial condition $S(0) = 6$. Reproduce this sketch below. Estimate the amount of salt in the tank after 15 minutes.

25 lbs of salt



The differential equation you developed for the salty tank is not separable, and therefore using the technique of separation of variables is not appropriate. This differential equation is called **first order linear**, which means it has the form

$$\frac{dy}{dt} + g(t) \cdot y = r(t),$$

where $g(t)$ and $r(t)$ are both continuous functions.

The following technique, which we refer to as the **reverse product rule**, can be used find the general solution to a first-order linear equation.

2. Review the product rule as you remember it from calculus. In general symbolic terms, how do you represent the product rule? How would you describe it in words?

$$(fg)' = f'g + g'f$$

Consider the differential equation $\frac{dy}{dt} + 2y = 3$. Note that this is a first order linear differential equation, where $g(t)$ and $r(t)$ are both continuous functions. The following illustrates a technique for finding the general solution to linear differential equations. The inspiration for the technique comes from a creative use of the product rule and the Fundamental Theorem of Calculus, as well as use of the previous technique of separation of variables.

Use the product rule to expand $(yu)'$.	$y'u + u'y$	Box 0
In the equation $\frac{dy}{dt} + 2y = 3$, rewrite $\frac{dy}{dt}$ as y'.	$y' + 2y = 3$	Box 1
Notice that the left-hand side of the equation in Box 1 looks a lot like the expanded product rule but is missing the function u. So multiply both sides by u, a function that we will determine shortly.	$uy' + 2uy = 3u$	Box 2
Because, so far, u is an arbitrary function, we can have u satisfy any differential equation that we want. Use $u' = 2u$ to rewrite the left-hand side of Box 2 to look like Box 0.	$u' = 2u$ $uy' + u'y = 3u$	Box 3

Use separation of variables to solve $u' = 2u$.	Box 4 $\int \frac{du}{u} = \int 2\,dt \Rightarrow \ln u = 2t + C$ $u = e^{2t}$ (omit C)
Replace u in the equation from Box 2 with your solution from Box 4.	Box 5 $e^{2t}y' + 2e^{2t}y = 3e^{2t}$
Show that the equation in Box 5 can be rewritten as $\left(ye^{2t}\right)' = 3e^{2t}$ *Hint*: Consider Box 0.	Box 6 product rule $(ye^{2t})' = 3e^{2t}$
Write integrals with respect to t on both sides. Apply the Fundamental Theorem of Calculus.	Box 7 $\int (ye^{2t})'\,dt = \int 3e^{2t}\,dt$
Obtain an explicit solution by isolating $y(t)$.	Box 8 $ye^{2t} = \frac{3}{2}e^{2t} + C$

3. Use the previous technique, which we refer to as the **reverse product rule**, to find the general solution for the Salty Tank differential equation from Problem 1. = integrating factors

$$\frac{dS}{dt} + \frac{1}{15+t}S = 2$$

$$\mu = e^{\int \frac{1}{15+t}dt} = e^{\ln(15+t)} = 15+t$$

$$(15+t)\frac{dS}{dt} + S = 2(15+t)$$

$$\int \left[(15+t)S\right]'dt = \int 2(15+t)\,dt$$

$$(15+t)S = 2\left(15t + \tfrac{1}{2}t^2\right) + C$$

$$S = \frac{30t+t^2}{15+t} + \frac{C}{15+t}$$

4. (a) Use the general solution from problem 3 to find the particular solution corresponding to the initial condition $S(0) = 6$ and then use the particular solution to determine the amount of salt in the tank after 15 minutes. That is, compute $S(15)$. Your answer should be close to your estimate from problem 1c. Is it? If not, you likely made an algebraic error.

$$6 = \cancel{\frac{30(0)+(0)^2}{15+0}}^{\to 0} + \frac{c}{15+0}$$

$c = 90$

$S(15) = 25.5$

it is close!

$$S = \frac{30t + t^2 + 90}{15+t} = \frac{t^2+30t+90}{t+15}$$

(b) What does your solution predict about the amount of salt in the tank in the long run? How about the concentration?

it will increase w/o bound, as long as the tank is big enough

the concentration will approach the concentration of the incoming brine

$$\begin{array}{r} t+15 \\ t+15 \overline{)\, t^2+30t+90} \\ t^2+15t \\ \hline 15t+90 \\ -\ 15t+225 \\ \hline -135 \end{array}$$

(c) Explain how you can make sense of the predictions from 4b by using the differential equation itself.

$$\frac{dS}{dt} = 2 - \frac{S}{15+t}$$

as this term $\to \infty$ in denom the fraction goes to 0

So $\frac{dS}{dt} \to 2$ over time

but as tank fills, the amount of salt goes up, but concentration is relative to amount of salt and it, too stabilizes