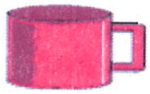
Cooling Coffee



A group of students want to develop a rate of change equation to describe the cooling rate for hot coffee in order that they can make predictions about other cups of cooling coffee. Their idea is to use a temperature probe to collect data on the temperature of the coffee as it changes over time and then to use this data to develop a rate of change equation.

The data they collected is shown in the table below. The temperature C (in degrees Fahrenheit) was recorded every 2 minutes over a 14 minute period.

Time (min)	Temp. (°F)
0	160.3
2	120.4
4	98.1
6	84.8
8	78.5
10	74.4
12	72.1
14	71.5

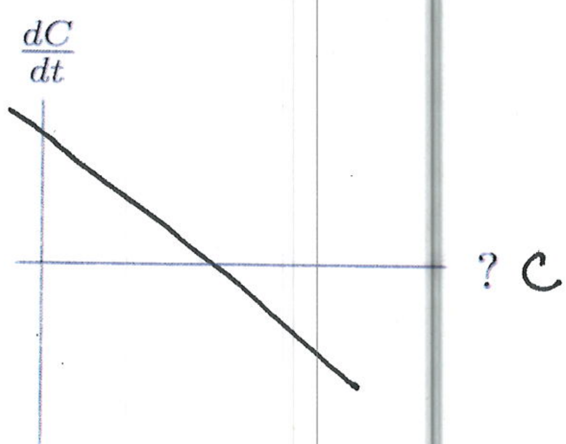
ΔT	T (midpoint)
-19.95	140.35
- 11.15	109.25
- 6.65	91.45
- 3.15	81.65
- 2.05	76.45
- 1.15	73.25
- 0.3	71.8

1. Figure out a way to use this data to create a third column whose values approximate $\frac{dC}{dt}$, where C is the temperature of the coffee.

2. Do you expect $\frac{dC}{dt}$ to depend on just the temperature C, on just the time t, or both the temperature C and the time t?

just C

3. Sketch below your best guess for the graph of $\frac{dC}{dt}$.



4. (a) Input the data from your extended table in question 1 into the GeoGebra applet https://ggbm.at/uj2gbz3V to plot points for $\frac{dC}{dt}$ vs. C. Does this plot confirm or reject your sketch from question 3?

answers will vary

(b) Toggle on the curve fitting tool and find an equation that fits your data.

$$\frac{dC}{dt} = -.283C + 19.7088$$

$$-.283(C - 69.622)$$

answers will vary somewhat

5. One group of students figured out that a reasonable rate of change equation to be

$$\frac{dC}{dt} = -0.4C + 28$$

which they rewrote as

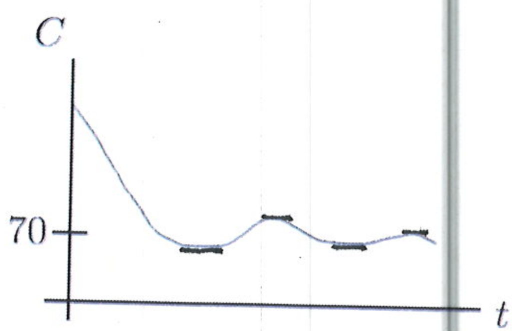
$$\frac{dC}{dt} = -0.4\,(C - 70)\,.$$

Interpret the meaning of the number 70 in this equation. Does this rate of change equation also make sense for predicting the future temperature of a glass of ice tea? Why or why not?

70 is room temperature

yes, since if coffee is higher than 70, eq. results in a negative $\frac{dC}{dt}$, but if temp is lower than room temp, one gets a positive $\frac{dC}{dt}$ which will be a warming of ice tea.

6. According to the rate of change equation from questions 4 and 5, is it possible for a graph of the **exact** solution to look like the one below? Why or why not? Give more than one reason for your answer.



No. this graph has multiple changes of direction, and thus multiple 0's. This is not possible for a linear ODE

also rate of change decreases more slowly so that it never achieves a min or max (or room temperature)

Population Growth - Limited Resources

A group of biologists want to study the population growth of certain bacteria in a laboratory. The scientists realized that the culture for the bacteria does not provide unlimited resources. Hence, the rate of change equation $\frac{dP}{dt} = kP$ is not appropriate. They conducted experiments to determine how the rate of change of population depends on just the population. The data they collected is shown in the table below (numbers are properly scaled). At various population levels, the scientists measured the population after one day.

Beginning Population	Population after one day
2	2.34
4	4.54
6	6.62
8	8.58
10	10.40
12	12.10
14	13.66
16	15.10
18	16.42
20	17.60

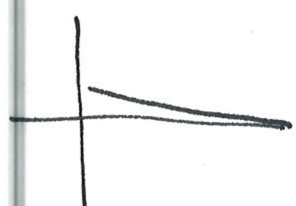
$\Delta P/\Delta t$
1.1
1.04
.98
.91
.85
.78
.72
.66
.59

7. Create a third column whose values approximate $\frac{dP}{dt}$. Explain why the method you used to create this column makes sense.

$$\frac{dP}{dt} \approx \frac{\Delta P}{\Delta t}$$

8. In this course we will call a graph of $\frac{dP}{dt}$ vs. P, when $\frac{dP}{dt}$ is an autonomous differential equation, an **Autonomous Derivative Graph**. Create an **autonomous derivative graph** and figure out a way to analyze this graph to determine the long term behavior for each of the beginning populations given in the table above.

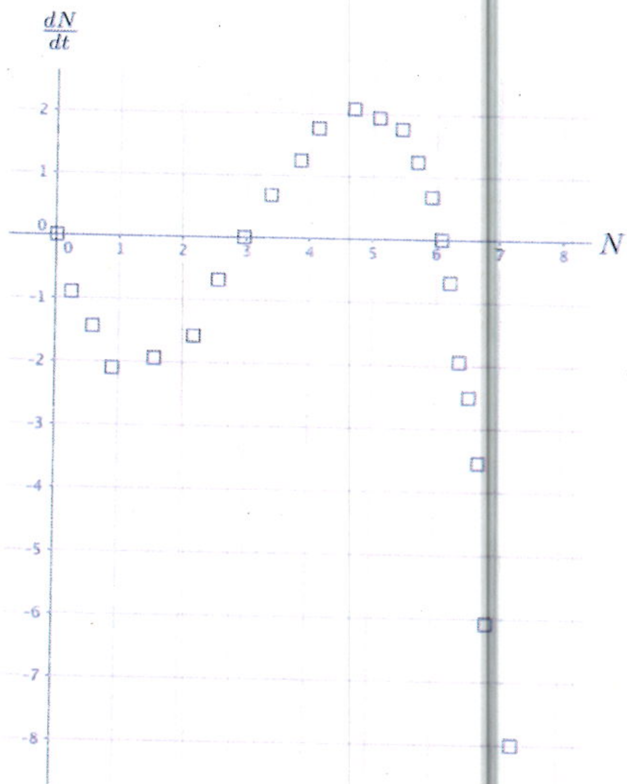
$$\frac{dP}{dt} = -.036P + 1.2069$$

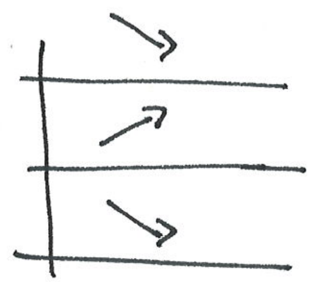


Zero at $X \approx 33.5$
carrying capacity

Analyzing Graphs of Autonomous Differential Equations

9. A group of biologists is studying a particular bug population in a rainforest. They gathered data about these bugs for different population values, N, at different times, t. The scientists reasoned that the rate of change depended only on the population and not on time. They approximated the derivatives $\frac{dN}{dt}$ (as was done with the cooling coffee from before) and plotted the autonomous derivative graph, as seen below:





For each part below, use the autonomous derivative graph to predict what the ultimate fate of the population will be. Describe (in words) the long-term behavior of each solution corresponding to the given initial condition. In addition, illustrate your conclusions with a suitable graph or graphs and classify all equilibrium solutions as either an attractor, repeller, or node.

(a) $N(0) = 2$ Slope negative ↓ decrease

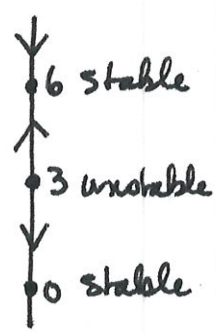
(b) $N(0) = 3$ stable / equilibrium

(c) $N(0) = 4$ Slope positive ↑ increase

(d) $N(0) = 4.5$ Slope positive ↑ increasing as fast as possible

(e) $N(0) = 6$ equilibrium

(f) $N(0) = 8$ Slope negative ↓ decreasing



10. Below is an autonomous derivative graph. Figure out the long-term behavior of every possible solution function and illustrate your conclusions with a suitable graph or graphs.