2/20/2019

Partial derivatives are derivatives on functions of more than one variable. Essentially, though, we are just going to look at one variable at a time.

The ordinary derivative is:

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

For the partial derivatives, we have to specific which variable we are concentrating on, since there is more than one. f_x is the derivative with respect to x, and f_y is the derivative with respect to y.

$$f_x = \lim_{h \to 0} \frac{f(x+h, y) - f(x, y)}{h}$$

$$f_y = \lim_{h \to 0} \frac{f(x, y+h) - f(x, y)}{h}$$

Notice that in each case, we are approaching the point (x, y) by only changing x or y but not both. In the end, all of our one-variable formulas still apply, but we pretend one variable is a "constant" and take the derivative of the one variable we care about.

Consider some examples.

$$f(x, y) = x^2 + xy - y^3 + 6x + 8y - 11$$

$$f_x = 2x + y + 6$$
$$f_y = x - 3y^2 + 8$$

For a term like x^2 if you are taking the derivative for x, do it normally. If you are taking the derivative for y then since there is no y in the expression, then treat it just like you would the -11: it's constant in the variable in question, and so the derivative is 0. If there is both x and y in the equation, take the derivative just like you would $6x$ or $8y$. The other variable is "constant".

For instance: $\frac{\partial}{\partial x}[xy] = y\frac{\partial}{\partial x}[x] = y(1) = y$ and $\frac{\partial}{\partial y}[xy] = x\frac{\partial}{\partial y}[y] = x(1) = x$.

A note about notation. When we do ordinary derivatives the notation $\frac{d}{dx}$ is the *operator* that says "take the derivative with respect to x for the function that follows". We use the ordinary letter d in this notation when there is only one variable in the equation. When there is more than one variable, we use a character called "del" = ∂, which is a modification of the Greek letter delta δ. This just indicates that there is more than one variable in the equation, but we are focusing on just one of them. So $\frac{\partial}{\partial x}$ is just the same as $\frac{d}{dx}$ and means "take the derivative with respect to x", but the function that follows has x and at least one additional variable.

We refer to these as "partial derivatives". So, f_x and f_y are first partial derivatives. And just like ordinary derivatives, we can take more of them. For partial derivatives we can also change variables.

So, there are potentially 4 second partial derivatives: $f_{xx}, f_{xy}, f_{yx}, f_{yy}$. The subscripts just tells us which variables we used in which order. So, f_{xx} says we took the derivative for x and then we took the derivative for x again. While f_{xy} says we started the same way with the derivative for x, but then, for the second derivative we took with respect to y.

Example:

$$f(x,y) = x^2y + e^{3x} + \ln(x+y) + 1$$

$$f_x = 2xy + 3e^{3x} + \frac{1}{x+y}$$
$$f_y = x^2 + \frac{1}{x+y}$$

$$f_{xx} = 2y + 9e^{3x} - \frac{1}{(x+y)^2}$$
$$f_{xy} = 2x - \frac{1}{(x+y)^2}$$
$$f_{yx} = 2x - \frac{1}{(x+y)^2}$$
$$f_{yy} = -\frac{1}{(x+y)^2}$$

Notice that $f_{xy} = f_{yx}$. This will be true all the time.

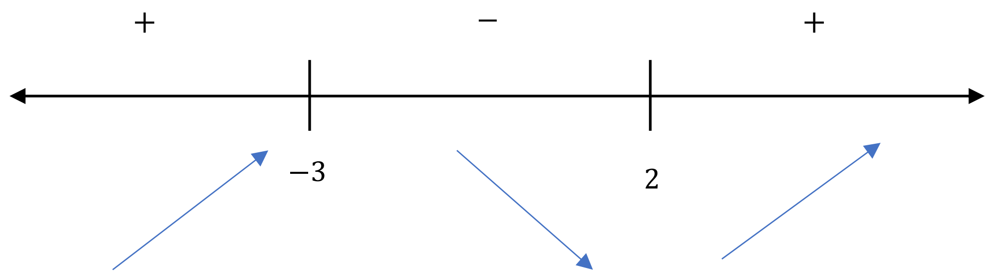
First Derivative Test/Optimization

The first derivative gives us information on the rate of change, or the slope of the tangent line. We can use this information to find any points that are relative maxima (points bigger than all nearby points) or relative minima (points smaller than all nearby points). These points are generally called extrema or critical points. They happen when the derivative is 0, or undefined. Maxima have slopes that form /\ (increasing, then change to decreasing), and minima have slope that form \/ (decreasing then switch to increasing). To test critical points, we usually make a sign chart.

Consider the function $f(x) = 2x^3 + 3x^2 - 36x + 11$.

Take the derivative. $f'(x) = 6x^2 + 6x - 36$. To find the critical points, set the derivative equal to 0. $6x^2 + 6x - 36 = 0$. Factor out 6. $6(x^2 + x - 6) = 0$. Factor the rest. $6(x-2)(x+3) = 0$. This will be 0 when $x = 2, x = -3$.

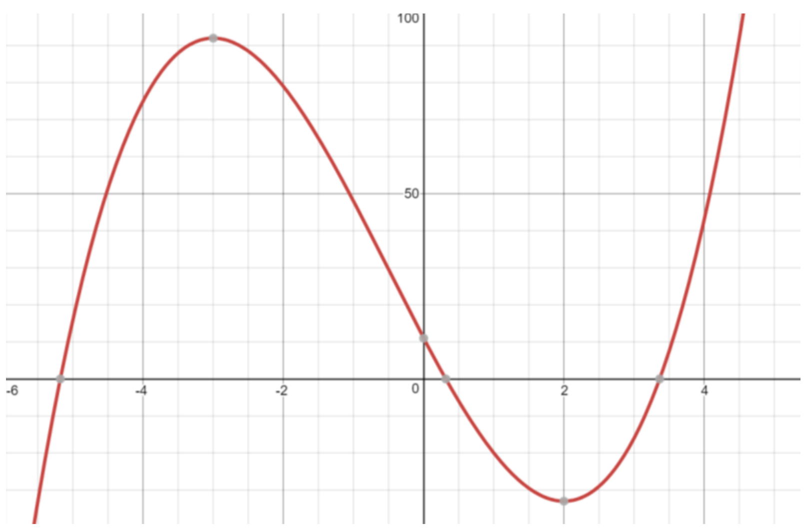
Draw the number line and test points in each section to get the signs of the derivative.



Check $f'(-4)$ on the first section, $f'(0)$ in the middle, and $f'(3)$ for the one on the right.

$$6(-4-2)(-4+3) = +(-)(-) = +$$
$$6(0-2)(0+3) = +(-)(+) = -$$
$$6(3-2)(3+3) = +(+)(+) = +$$

We don't really care about the value here, just the signs. Where the derivative is positive, the graph is increasing. Where the derivatives is negative, the graph is decreasing. So we can see that $x = -3$ is a maximum, and $x = 2$ is a minimum. If we plug the two critical points into the original function and plot them, we can get a pretty accurate sketch of the graph.



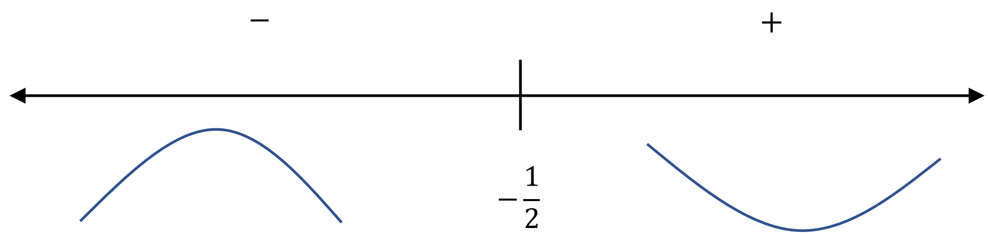
You can see from the graph that the relative maximum is at $x = -3$ and the relative minimum is at $x = 2$.

The Second Derivative Test

The second derivative does give increasing or decreasing information; instead, it gives us information on whether the graph is curved upward or downward (called concavity). If the second derivative is positive then the curve is upward like ∪, but if the second derivative is negative, the graph is curved downward, like ∩. When the second derivative is 0, this is called an inflection point, and it's where the graph changes from curving upward to curving downward.

Consider our function above: function $f(x) = 2x^3 + 3x^2 - 36x + 11$.

Take the derivative. $f'(x) = 6x^2 + 6x - 36$. And take the second derivative $f'' = 12x + 6$. The second derivative is 0 at $12x + 6 = 0 \rightarrow x = -\frac{6}{12} = -\frac{1}{2}$. We can also draw a number line for this as well.



If we check points on each segment, we can check $f''(-1)$ and $f''(0)$. Again, we care basically just about the sign.

$$f''(-1) = 12(-1) + 6 = -$$
$$f''(0) = 12(0) + 6 = +$$

The second derivative test says that if the critical point produces a negative result in the second derivative, then it is a maximum, and we can see why this would be the case, since concave down, means there is a top of the curve. If the critical point is positive, then the curve is concave up, and so the critical point is the bottom of the curve.

$$f''(-3) = 12(-3) + 6 = -12$$
$$f''(2) = 12(2) + 6 = 36$$

And we can see that this agrees with our graph.