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Polynomials

$$P(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_2 x^2 + a_1 x + a_0$$

Leading term – term with the highest power of x

$$f(x) = 3x^4 + 2x^3 - x^2 + x - 11$$

The leading term in $3x^4$

Descending order has the highest-powered term first, and then descends in power to the constant

Leading coefficient is the coefficient of the highest-powered term = 3

The leading coefficient = a_n

The constant term = $a_0 = a_0 x^0$ because $x^0 = 1$

End behavior of the polynomial

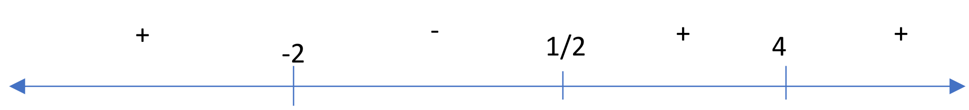
Odd-powered polynomials: when $x \to -\infty, y \to -\infty$ and when $x \to \infty, y \to \infty$ (switches when sign of leading coefficient change)
Even-powered polynomials (other than n=0): when $x \to \pm\infty, y \to \infty$ (when sign of the leading coefficient changes, then $y \to -\infty$)

Degree: for a polynomial of a single variable, the degree of the polynomial is the same as the degree of the highest-powered term (degree itself is that same power).

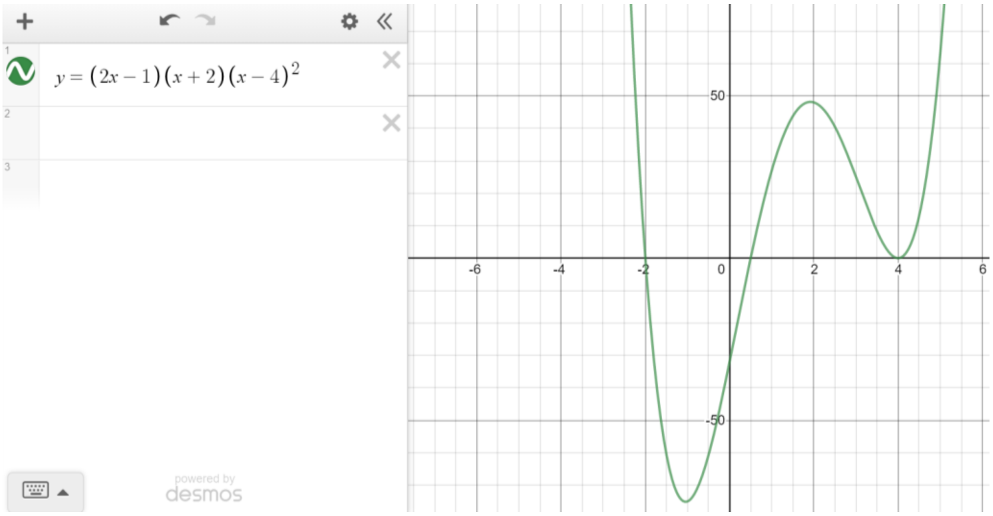
The constant function is technically a polynomial $f(x) = a_0$. This is said to be a polynomial of degree 0, unless $a_0 = 0$. Then the polynomial is said to have "no degree".

Intermediate Value Theorem
For a smooth function, if $f(a) < 0$, and $f(b) > 0$ then there is a zero between $x = a$ and $x = b$. In other words, there is a point on the interval (a, b) or (b, a) where $f(c) = 0$, and $c \in (a, b)$.

Constructing Sign Diagrams for general polynomials
1) Find the zeros (factoring, finding them numerically from a graph, etc.)
2) Place the zeros on a number line in order from smallest to largest
3) Test points in each interval



$$f(x) = (2x - 1)(x + 2)(x - 4)^2$$



Multiplicity:

$$(x-4)^2 = (x-4)(x-4)$$
$$(x-4)^3 = (x-4)(x-4)(x-4)$$

Multiplicity is just the number of copies in the factored polynomial of a given factor.

Multiplicity 1 factors (only one copy) near the zero is going to behave like a linear equation: goes through the zero for that factor in more-or-less a straight line; there is a sign change

Multiplicity 2 factors (there are two copies, the factor is squared) is going to behave like a quadratic near the zero: approach zero, and then "bounce" off the axis and remain the same sign on both sides of the zero

Multiplicity 3 factors (there are three copies, the factor is cubed) is going to behave like a cubic near the zero: it will cross through the axis, but with a kink

Higher multiplicities: even ones behave like the multiplicity 2 (but flatter), and odd ones behave like the multiplicity 3 (but flatter).

Division of polynomials & Synthetic division

Factor the polynomial: $f(x) = x^3 + 4x^2 - 5x - 14$
Zero of the polynomial at x=2
Corresponding factor $x-2$=0
$\frac{x^3+4x^2-5x-14}{x-2}$= missing quadratic factor

$$\begin{array}{r}
x^2+6x+7 \\
x-2\,|\overline{x^3+4x^2-5x-14} \\
\underline{-(x^3-2x^2)} \\
6x^2-5x-14 \\
\underline{-(6x^2-12x)} \\
7x-14 \\
\underline{-(7x-14)} \\
0
\end{array}$$

$$f(x) = x^3 + 4x^2 - 5x - 14 = (x-2)(x^2 + 6x + 7)$$

This polynomial does not factor further. If we need the other zeros, we can use the quadratic formula to find them.

Polynomial long division like this works for all polynomials and all polynomial divisors (assuming the degree of the numerator is larger than the degree of the denominator (factor)). Synthetic division only works for linear factors $(x - c)$, where c is the zero.

For a factor like $2x - 1$, you would have to write this as $2\left(x - \frac{1}{2}\right)$ and then use synthetic division on the $x - \frac{1}{2}$.

Synthetic division:

$$\begin{array}{r|lllll} c & a_n & a_{n-1} & \ldots & a_2 \; a_1 & a_0 \\ & & ca_n & & & \\ \hline & a_n & & & & \end{array}$$

$$\begin{array}{r|rrrr} 2 & 1 & 4 & -5 & -14 \\ & & 2 & 12 & 14 \\ \hline & 1 & 6 & 7 & \color{red}{0} \end{array}$$

1. Write out the coefficient of the polynomial on the right side of the bar, and the zero of the linear factor on the left side.
2. Skip a line, and bring the leading coefficient down to line 3
3. Multiply the zero by the leading coefficient and put on line 2 under the next term
4. Add.
5. Repeat steps 3 and 4 until you get to the end.

Two very important theorems that stem from what we've just done:
The Remainder Theorem & The Factor Theorem

The Remainder Theorem: If you can divide a polynomial by a factor (x-c), and get a 0 remainder, then the c is a zero of the polynomial.

Is that the remainder of polynomial division for a polynomial p(x) by $x - c$, is the same as the value of $p(c)$.

The Factor Theorem: If you can divide a polynomial by another polynomial q(x), and get zero remainder, then the polynomial q(x) is a factor of the original polynomial, and the quotient of the division is the other factor.

If the polynomial p(x)=0 at some value c, then c is a zero of the polynomial, and $x - c$ is a factor (if you divide it out, you will get a remainder of zero).

We are trying to find out what the zeros of a polynomial are without having a calculator.

Cauchy Bounds: If you divide out the leading coefficient, then the remaining largest coefficient (in absolute value) is the bound on the zeros of the polynomial.

$$f(x) = x^3 + 4x^2 - 5x - 14$$

Coefficients: 1, 4, -5, -14. The largest absolute value is 14. So there can be no zeros outside $[-14,14]$.

Set the margins on your calculator to [-15,15].

Rational Zeros Theorem (zeros that can be written as a fraction: not including square roots)

List the factors of a_n (leading coefficient) q_i
List the factors of a_0 (constant term) p_j

The possible rational zeros can only be: $\pm\frac{p_j}{q_i}$

Polynomial $2x - 3$

$$a_n = 2, a_0 = 3$$

$$q_i: 1, 2$$
$$p_j: 1, 3$$

Possible rational zeros can only be $\pm\frac{1}{1}, \pm\frac{1}{2}, \pm\frac{3}{1}, \pm\frac{3}{2}$

$$2x - 3 = a_n x + a_0 \rightarrow a_n x = -a_0 \rightarrow x = -\frac{a_0}{a_n}$$

The same list would apply to $2x^2 - 3,\ 2x^2 - x + 3,\ 2x^3 - x^2 + 5x - 3,\ etc.$

Bounds on zeros is that the largest rational zero in this list is ± 3, so this can also act as a bounds on the zeros.

Can set graph range to [-4,4].

Descartes' Rule of Signs

1. Count the number of sign changes in your polynomial
2. The number corresponds to the number of positive real zeros (P, P-2, P-4…)
3. Replace x with -x to get $p(-x)$
4. Count the sign changes in the resulting polynomial
5. The number of negative real zeros (N, or N-2, N-4…)

$$f(x) = x^3 + 4x^2 - 5x - 14$$

Has one sign change: there must be one positive real zero

$$f(-x) = -x^3 + 4x^2 + 5x - 14$$

Has two sign changes: there must be either 2 negative zeros, or no negative zeros.

This means that if we find the one positive zero, then we can discard all other positive possible solutions.

Complex zeros

Fundamental Theorem of Algebra:
For a polynomial of degree n, there are exactly n linear factors (counting multiplicity), and all allowing for complex zeros.

…There are exactly n zeros (real or complex), counting multiplicities.

Real Factorization Theorem:
Every (real coefficients) polynomial can be factored as real linear factors, and unfactorable quadratics with real coefficients.

In any polynomial with real coefficients, all complex zeros must occur in conjugate pairs.

$$z = 3 + 4i$$
$$\bar{z} = 3 - 4i$$
$$|z| = \sqrt{9 + 16} = 5$$

If $a + bi$ is a zero of real-coefficient polynomial, then so is $a - bi$.

Complex Number Revie

$$i = \sqrt{-1}, i^2 = -1$$
$$(4 + 3i) + (-2 + 5i) = (4 - 2) + (3i + 5i) = 2 + 8i$$

$$(4 + 3i)(-2 + 5i) = -8 + 20i - 6i + 15i^2 =$$
$$-8 + 20i - 6i + 15(-1) = -23 + 14i$$

$$\frac{-2 + 5i}{4 + 3i} \times \frac{4 - 3i}{4 - 3i} = \frac{-8 + 6i + 20i - 15i^2}{16 - 12i + 12i - 9i^2} = \frac{7 + 26i}{16 + 9} = \frac{7}{25} + \frac{26}{25}i$$

$$\|z\| = |z| = \sqrt{a^2 + b^2}$$

Find the quadratic that represents the zeros $3 \pm 4i$

$$\big(x - (3 + 4i)\big)\big(x - (3 - 4i)\big) = (x - 3 - 4i)(x - 3 + 4i) =$$
$$x^2 - 3x + 4ix - 3x + 9 - 12i - 4ix + 12i - 16i^2 =$$
$$x^2 - 6x + 25$$

A quadratic with real coefficients