3/04/2021

Rational Functions

Definition of a rational function: $\frac{p(x)}{q(x)}$ where $p(x), q(x)$ are polynomials and $q(x)$ is not identically zero.

Domain and range:
Domain is the values of x where $q(x) \neq 0$.

$$r(x) = \frac{3x+1}{x^2 - x - 2}$$

$$x^2 - x - 2 = 0$$
$$(x-2)(x+1) = 0$$
$$x = 2, -1$$

Domain: $(-\infty, -1) \cup (-1, 2) \cup (2, \infty)$ or $\{x | x \neq -1, 2\}$

Range of a rational function is the domain of the inverse of the function. (The domain and range switch roles.)

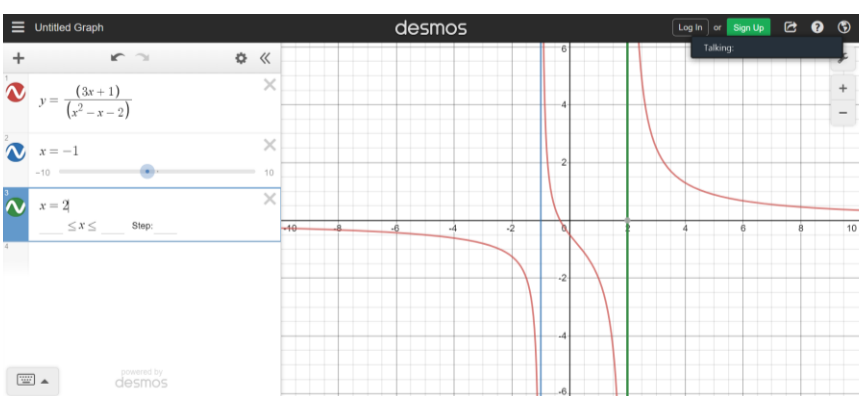
Horizontal asymptotes may be the hole in the range:
The degree of the denominator is larger than the degree of the numerator, and so the horizontal asymptote could be $y = 0$. The range is $(-\infty, 0) \cup (0, \infty)$. Check the graph to be sure.

Is there a value of x that will make $r(x) = 0$?

$$0 = \frac{3x+1}{x^2 - x - 2}$$

$$0 = 3x + 1$$
$$3x = -1$$
$$x = -\frac{1}{3}$$

So, we can say that the range is all real numbers.



Vertical Asymptotes:
Vertical lines that the function approaches but cannot cross (discontinuity) that occurs when the denominator is equal to zero.*

*In factored form, these are zeros of the denominator that do not cancel with anything in the numerator. Zeros of the reduced form of the rational function.

When the denominator is zero, but the factor cancels with the same factor in the numerator, you get a hole.

$$r(x) = \frac{x^2 - 5x + 6}{x^2 - 2x - 3}$$

$$r(x) = \frac{(x-3)(x-2)}{(x-3)(x+1)}$$

$$r_{reduced}(x) = \frac{x-2}{x+1}$$

Domain: $x \neq -1{,}3$, but $x = -1$ is a vertical asymptote, and $x = 3$ is a hole.
Holes don't plot on the graph in your calculator, but you must show them on your hand-drawn graph.

The hole is plotted with an open circle on the reduced graph. What is the y-coordinate of the hole? Plug the value into the reduced graph.

$$r_{reduced}(3) = \frac{3-2}{3+1} = \frac{1}{4}$$

Coordinate of the hole: $\left(3, \frac{1}{4}\right)$.

Horizontal asymptotes
Occur when a) the denominator has a higher degree than the numerator, in which case the HA is $y = 0$, b) the denominator and the numerator have the same degree, then the HA is result of dividing the two leading terms.

Horizontal asymptotes are always constants.
Case c) the numerator is a higher degree than the denominator, then you need to do long division to obtain the "slant" asymptote.

$$r(x) = \frac{x-2}{x+1}$$

The leading term in the numerator is x. The leading term in the denominator is also x. $\frac{x}{x} = 1$. So the horizontal asymptote is $y = 1$.
This is a horizontal line that, for large values of x, the graph will approach but never cross. (The graph may cross the line for small values of x (near 0).)

Slant or Oblique Asymptotes (are more other general asymptote shapes we won't cover)
Occurs when the numerator is one degree higher than the degree in the denominator.

$$r(x) = \frac{x^2 + 6x + 8}{x + 3}$$

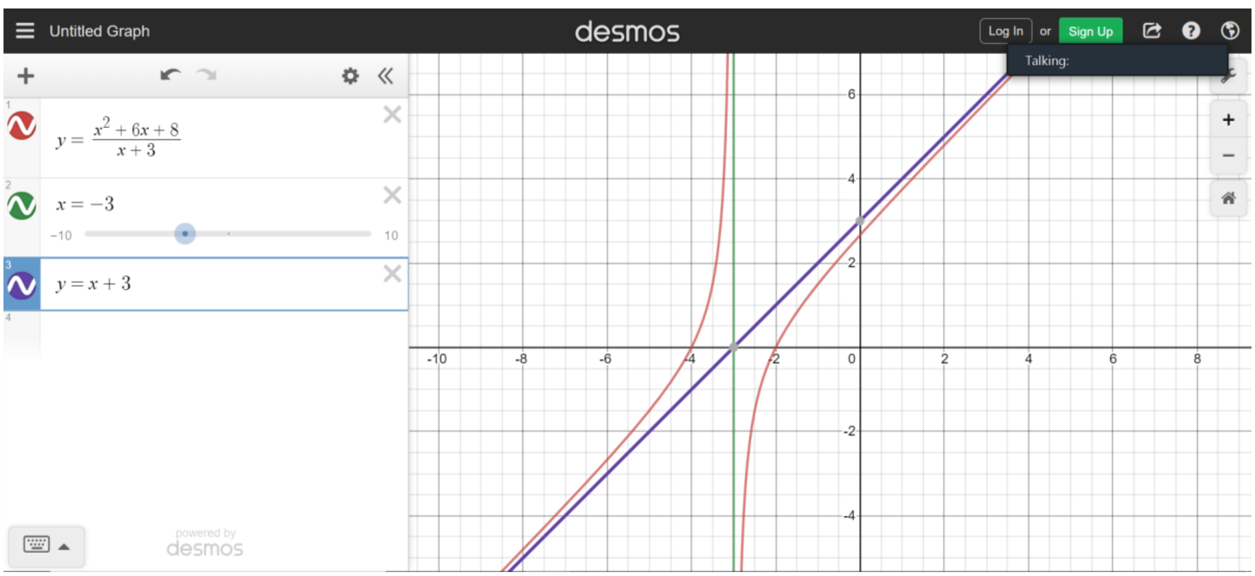
Domain: $x \neq -3$, Vertical asymptote is at x=3.
Slant asymptote: get from long division (synthetic division if the denominator is linear)

$$\begin{array}{r|rrr} -3 & 1 & 6 & 8 \\ & & -3 & -9 \\ \hline & 1 & 3 & -1 \end{array}$$

$$r(x) = x + 3 - \frac{1}{x + 3}$$

$(x + 3)(x + 3) = x^2 + 6x + 9$, then subtract 1 to get numerator

Slant asymptote is $y = x + 3$ (the non-remainder result of the division)



How to analyze the graph of a rational function. How to plot a rational function by hand.

Step 1. Find the domain.
Step 2. Factor the numerator (and denominator) and reduce the expression.
Step 3. Identify the vertical asymptotes (factors from denominator that do not cancel), and the holes (factors that did cancel). Draw both on the graph.
Step 4. Determine whether you have a horizontal or slant asymptote.
Step 5. Find and plot those asymptotes.
Step 6. Find and plot any intercepts (y and x)
Step 7. Plot additional points in each section of the graph as needed.
Step 8. Connect the dots.

Step 9. Now you can find the range.

$$r(x) = \frac{4x}{x^2 - 4} = \frac{4x}{(x+2)(x-2)}$$

Domain: $x^2 - 4 = 0 \rightarrow (x-2)(x+2) = 0, x = 2, -2$

$$(-\infty, -2) \cup (-2,2) \cup (2, \infty)$$

Cannot be reduced.
Vertical asymptotes are $x = 2, x = -2$
Horizontal asymptote is $y = 0$
Intercepts:

$$0 = \frac{4x}{x^2 - 4} \rightarrow 4x = 0 \rightarrow x = 0$$

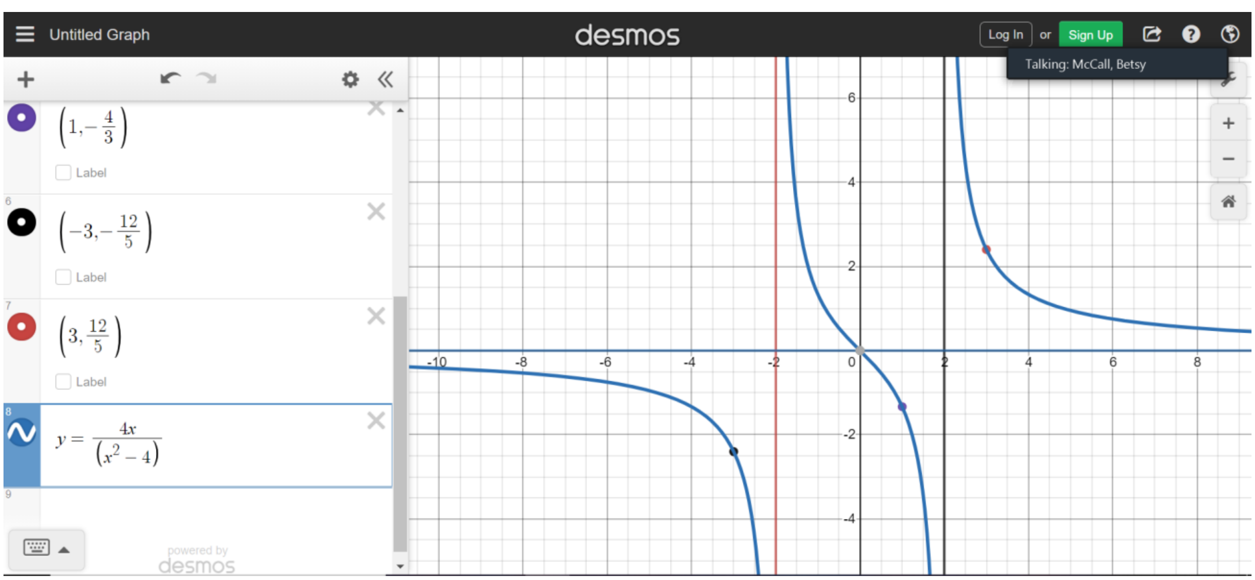
$$(0,0)$$

Additional points:

$$x = 1 \rightarrow \frac{4(1)}{1^2 - 4} = \frac{4}{-3} \rightarrow \left(1, -\frac{4}{3}\right)$$

$$x = -3 \rightarrow \frac{4(-3)}{[(-3)^2 - 4]} = -\frac{12}{5} \rightarrow \left(-3, -\frac{12}{5}\right)$$

$$x = 3 \rightarrow \frac{4(3)}{[(3)^2 - 4]} = \frac{12}{5} \rightarrow \left(3, \frac{12}{5}\right)$$



New example:

$$r(x) = \frac{x^2 - x - 6}{x + 1} = \frac{(x+2)(x-3)}{x+1}$$

Domain: $x \neq -1$

Nothing cancels
Vertical asymptote is $x = -1$

$$\begin{array}{r|rrr} -1 & 1 & -1 & -6 \\ & & -1 & 2 \\ \hline & 1 & -2 & -4 \end{array}$$

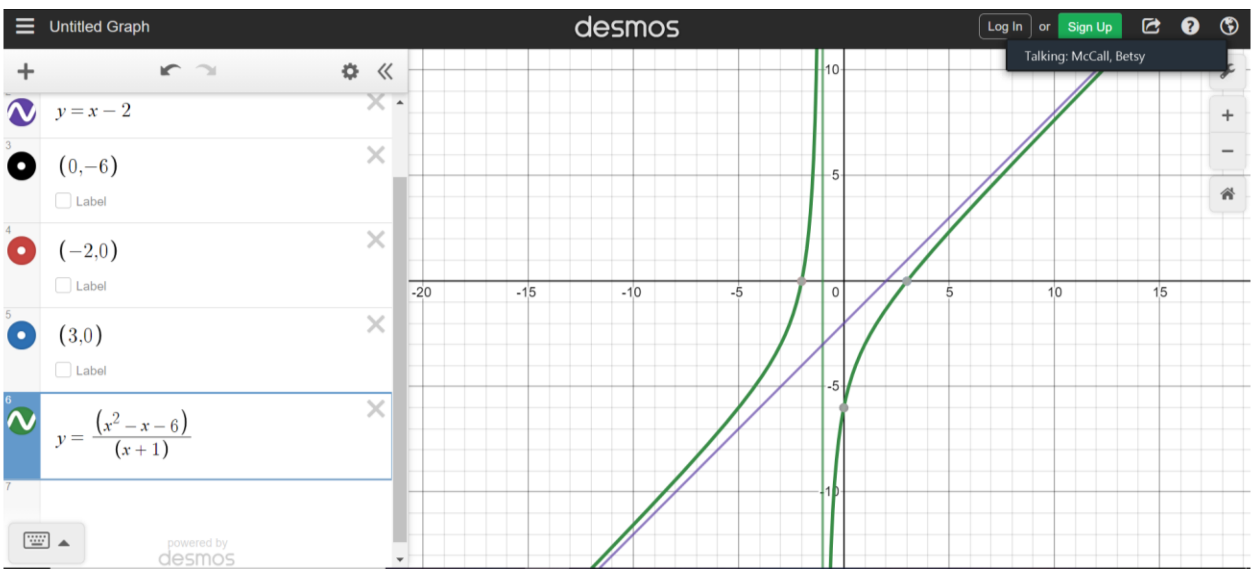
$$r(x) = x - 2 - \frac{4}{x+1}$$

Slant asymptote is $y = x - 2$

Intercepts:

$$r(0) = \frac{0^2 - 0 - 6}{0 + 1} = -\frac{6}{1} = -6 \to (0, -6)$$

$$0 = \frac{x^2 - x - 6}{x + 1} \to x^2 - x - 6 = 0 \to (x+2)(x-3) = 0 \to x = -2, x = 3 \to (-2,0), (3,0)$$



Repeated factors are going to behave slightly differently.

Some simple rational functions can be graphed with transformations.

Not all rational function have holes or vertical asymptotes.

Polynomial and Rational Inequalities

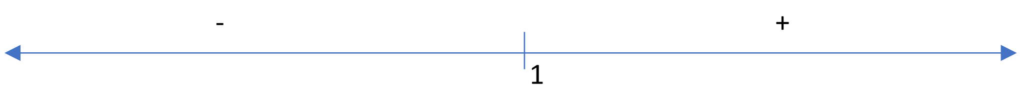
$$x^3 - 1 \geq 0$$
$$\frac{x+1}{x+2} \geq 1$$

And so forth.

Step 1. Put everything on one side of the equation and let the other side be 0.
Step 2. Simplify and write everything as a single expression (esp. for rationals).
Step 3. Factor everything. Find all the real zeros. For rationals, do this for both the numerator and the denominator.
Step 4. Put everything on the number line.
Step 4B. For rational expressions, it's helpful when working with inequalities that include an equal sign to mark which zeros came from the denominator.
Step 5. Make sign chart.
Step 6. Write the expression to solve the inequality.

$$x^3 - 1 \geq 0$$
$$(x-1)(x^2+x+1) \geq 0$$

Zero is at x=1



Test values: 0, 2
Solution: $[1, \infty)$

If this was

$$x^3 - 1 > 0$$

Solution: $(1, \infty)$

$$\frac{x+1}{x+2} \geq 1$$

$$\frac{x+1}{x+2} - 1 \geq 0$$

Find a common denominator to simplify.

$$\frac{x+1}{x+2} - \frac{1(x+2)}{x+2} \geq 0$$

$$\frac{x+1-(x+2)}{x+2} \geq 0$$

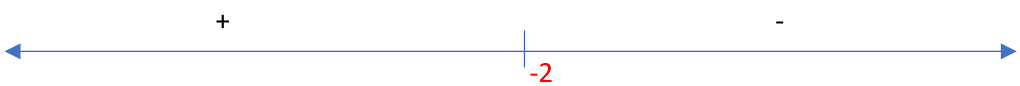
$$\frac{x+1-x-2}{x+2} \geq 0$$

$$-\frac{1}{x+2} \geq 0$$

If there are any x's left in the numerator, set equal to 0 to obtain the zero of the polynomial. Same for the denominator.

The only zero to worry about here is the zero from the denominator: $x + 2 = 0 \rightarrow x = -2$

Once you have all the zeros, plot them on a number line just like with the polynomial.



Red number(s) are from the denominator.

Test values: -3, 0

$$-\frac{1}{-3+2} = -\frac{1}{-1} = +$$
$$-\frac{1}{0+2} = -\frac{1}{2} = -$$

Solution: $(-\infty, -2)$
Zeros from the denominator can never have square brackets regardless of the inequality in the original problems.