04/01/2021

Exponentials and logarithms, continued

Examples.

$$2^{4x} = 8$$
$$2^{4x} = 2^3$$
$$4x = 3$$
$$x = \frac{3}{4}$$

$$8^x = \frac{1}{128} = \frac{1}{2^7} = 2^{-7}$$

$$(2^3)^x = 2^{-7}$$
$$2^{3x} = 2^{-7}$$
$$3x = -7$$
$$x = -\frac{7}{3}$$

$$5^x = 2$$

$$\log_a a^x = x$$
$$a^{\log_a x} = x$$

$$\log_5 5^x = \log_5 2$$
$$x = \log_5 2 = \frac{\ln 2}{\ln 5} \approx 0.4306765 \ldots$$

$$5^x = -2$$
No solution

$$70 + 90e^{-0.1t} = 75$$
$$90e^{-0.1t} = 5$$

$$e^{-0.1t} = \frac{1}{18}$$
$$\ln(e^{-0.1t}) = \ln\left(\frac{1}{18}\right)$$
$$-0.1t = \ln\left(\frac{1}{18}\right)$$

$$t = \frac{\ln\left(\frac{1}{18}\right)}{-0.1} \approx 28.9$$

$$e^{2x} - 3e^x - 10 = 0$$

(quadratic form) $(e^x)^2 - 3(e^x) - 10 = 0$

Substitute: $u = e^x$

$$u^2 - 3u - 10 = 0$$
$$(u+2)(u-5) = 0$$

$$u = -2$$
$$e^x = -2 \text{ no solution}$$

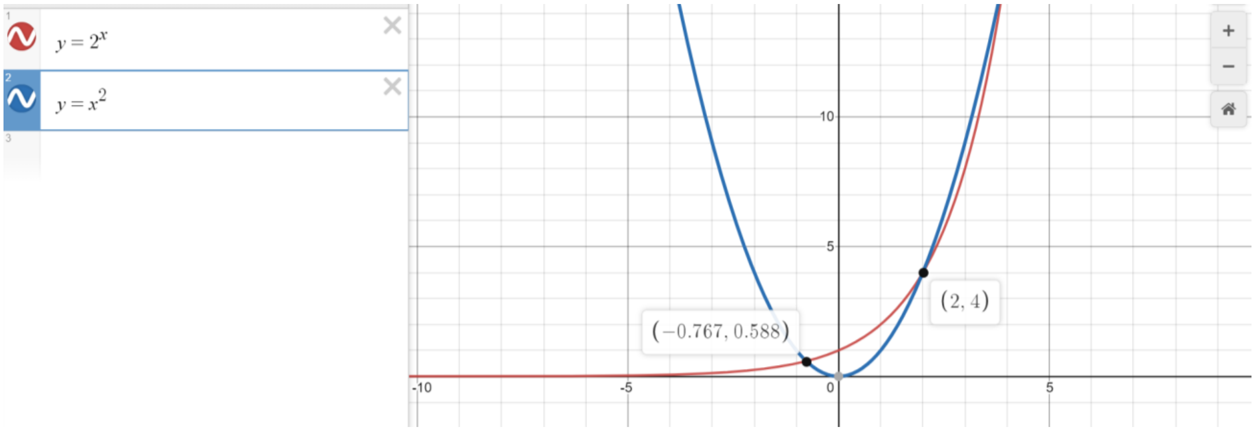
$$u = 5$$
$$e^x = 5$$
$$\ln(e^x) = \ln(5)$$
$$x = \ln 5$$

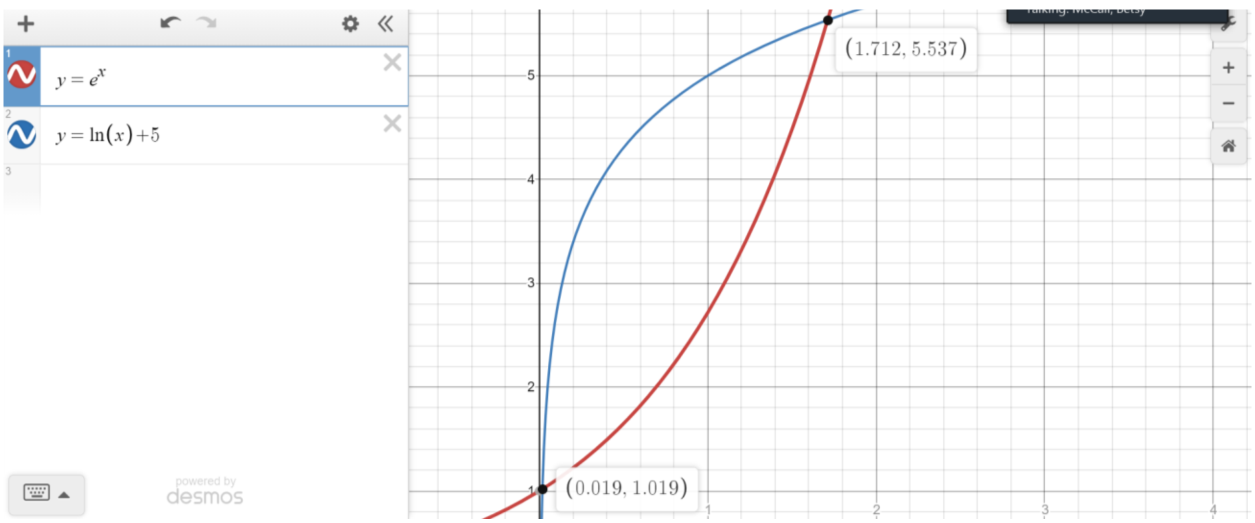
$$4^x + 2^x = 12$$
$$4^x + 2^x - 12 = 0$$
$$(2^2)^x + 2^x - 12 = 0$$
$$(2^x)^2 + 2^x - 12 = 0$$
$$u = 2^x$$
$$u^2 + u - 12 = 0$$
$$(u+4)(u-3) = 0$$
$$u = -4$$
$$2^x = -4 \text{ no solution}$$
$$u = 3$$
$$2^x = 3$$
$$\log_2 2^x = \log_2 3$$
$$x = \log_2 3 = \frac{\ln 3}{\ln 2}$$

$$7^{3+7x} = 3^{4-2x}$$
$$\ln(7^{3+7x}) = \ln(3^{4-2x})$$
$$(3+7x)\ln(7) = (4-2x)\ln(3)$$
$$3\ln 7 + 7\ln 7\,(x) = 4\ln 3 - 2\ln 3\,(x)$$
$$2\ln 3\,(x) + 7\ln 7\,(x) = 4\ln 3 - 3\ln 7$$
$$x[2\ln 3 + 7\ln 7] = 4\ln 3 - 3\ln 7$$
$$x = \frac{4\ln 3 - 3\ln 7}{2\ln 3 + 7\ln 7} \approx -0.091239 \ldots$$

Problems that can’t be solved by hand: equation that involve both exponential and logs, exponential/logs with bare variables

$$2^x = x^2$$
$$e^x = \ln(x) + 5$$





$$\log_2(x^3) = \log_2 x$$

$$\log_2 x^3 = \log_2(x^3)$$
$$(\log_2 x)^3 = \log_2^3 x$$

$$x^3 = x$$
$$x^3 - x = 0$$
$$x(x^2 - 1) = 0$$
$$x(x-1)(x+1) = 0$$
$$x = 0, 1, -1$$

$\log_2(0^3) = \log_2 0$ -- log of 0 is not defined

$\log_2(-1^3) = \log_2 -1$ -- log of negative numbers is not defined

$$x = 1$$

$$\log_3(7 - 2x) = 2$$
$$3^{\log_3(7-2x)} = 3^2$$

$$7-2x=9$$
$$-2x=2$$
$$x=-1$$

$$\log_3(7-2(-1))=\log_3(9)=2$$

$$\log_3(x-4)+\log_3(x+4)=2$$
$$\log_3[(x-4)(x+4)]=2$$

$$3^{\log_3[(x-4)(x+4)]}=3^2$$
$$x^2-16=9$$
$$x^2-25=0$$
$$(x-5)(x+5)=0$$
$$x=5,-5$$

$\log_3(5-4)+\log_3(5+4)$ good
$\log_3(-5-4)+\log_3(-5+4)$ bad – reject this solution

$$x=5$$

$$\log(x)-\log(2)=\log(x+8)-\log(x+2)$$
$$\log\left(\frac{x}{2}\right)=\log\left(\frac{x+8}{x+2}\right)$$
$$\frac{x}{2}=\frac{x+8}{x+2}$$
$$x(x+2)=2(x+8)$$
$$x^2+2x=2x+16$$
$$x^2-16=0$$
$$(x+4)(x-4)=0$$
$$x=4,-4$$

Check in the original problem
$\log(-4)$ bad

$$x=4$$

$$\ln(\ln x)=3$$
$$e^{\ln(\ln x)}=e^3$$
$$\ln x=e^3$$
$$e^{\ln x}=e^{e^3}$$
$$x=e^{e^3}\approx 5.28\times 10^8$$

$$(\log x)^2=2\log x+15$$

Quadratic form, $u=\log x$

$$u^2=2u+15$$
$$u^2-2u-15=0$$
$$(u-5)(u+3)=0$$
$$u=5,-3$$

$$\log x = 5$$
$$10^{\log x} = 10^5$$
$$x = 10^5 = 100{,}000$$

$$\log x = -3$$
$$10^{\log x} = 10^{-3}$$
$$x = 10^{-3} = 0.001$$

2 solutions

Applications

Simple Interest

$$A = P + Prt = P(1 + rt)$$

Compounded Interest

$$A = P\left(1 + \frac{r}{n}\right)^{nt}$$

P is the principal
r is the annual interest rate
t is time in years
n is the number of compounding periods per year (annually=1, quarterly=4, monthly=12, weekly=52, daily=365, etc.)

$300, earned 5% simple interest for 2 years.

$$A = 300(1 + 0.05 \times 2) = 300(1.1) = 330$$

$300, earned 5% interest compounded annually for 2 years.

$$A = 300\left(1 + \frac{0.05}{1}\right)^2 = 300(1.05)^2 = 330.75$$

$300, earned 5% interest compounded daily for two years

$$A = 300\left(1 + \frac{0.05}{365}\right)^{730} = 331.55$$

Compounding continuously (stock market average, inflation)

$$A = Pe^{rt}$$

$300, earned 5% continuously compounded interest for two years?

$$A = 300e^{0.5(2)} = 331.55$$

How long would it take for your initial investment to double?

$$600 = 300e^{0.05t}$$
$$2 = e^{0.05t}$$
$$\ln 2 = \ln e^{0.05t}$$
$$\ln 2 = 0.05t$$
$$\frac{\ln 2}{0.05} = t \approx 13.86 \ldots\ years$$

Exponential Growth and Decay

$$N(t) = N_0 e^{kt}$$

N_0 is the initial population
k is the (positive) growth rate, (or negative decay rate)
t is time

Newton's Law of Cooling

$$T(t) = T_a + (T_0 - T_a)e^{kt}$$

Logistic Growth

$$N(t) = \frac{L}{1 + Ce^{-kLt}} \qquad C = \frac{L}{N_0} - 1$$

Exponential decay example:
The half-life of carbon-14 is 5730 years.

$$100e^{k(5730)} = 50$$
$$e^{5730k} = \frac{1}{2}$$
$$\ln\left(e^{5730k}\right) = \ln\left(\frac{1}{2}\right)$$
$$5730k = \ln\left(\frac{1}{2}\right)$$
$$k = \frac{\ln\left(\frac{1}{2}\right)}{5730} \approx -0.0001209688 \ldots$$

$$N(t) = 100e^{-0.0001209688t}$$

End of Chapter 6

Exam #3 is next week!