04/15/2021

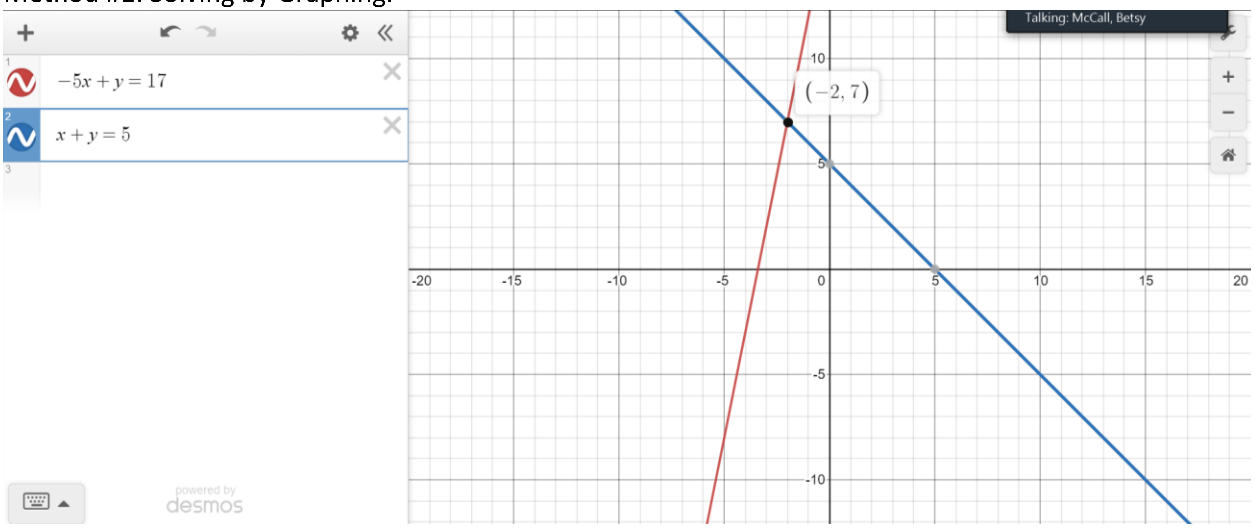
Systems of Equations and Matrices (Chapter 8)

Example in two-variables

$$\begin{cases} -5x + y = 17 \\ x + y = 5 \end{cases}$$

System of equations in two variables. The goal is to find a single pair of (x, y) that satisfies both equations.

Method #1: Solving by Graphing.



$$-5(-2) + 7 = 10 + 7 = 17$$
$$-2 + 7 = 5$$

Verified

In the calculator (TI) you will have to solve each equation for y in order to graph it.

Method #2: Solve by substitution.

Solve one equation for one variable and replace that variable in the other equation(s), then solve for the remaining variable.

$$\begin{cases} -5x + y = 17 \\ x + y = 5 \end{cases}$$

Pick a variable to solve for (if you can) that has a coefficient of 1 or -1.

$$y = 5 - x$$

Substitute into the top equation for y.

$$-5x + (5 - x) = 17$$

Solve for x.

$$-5x + 5 - x = 17$$

$$-6x = 12$$
$$x = -2$$

Plug this value into one of the equations to find the second variable.

$$y = 5 - (-2) = 7$$

Solution $(-2,7)$.

Method #3: Elimination by Addition

$$\begin{cases} -5x + y = 17 \\ \quad x + y = 5 \end{cases}$$

Goal: Modify one (or both) equations so that one of the variables is the same size by opposite sign as the same variable in the other equation, so that when you add the equation, one variable cancels.

$$\begin{cases} -5x + y = 17 \\ -x - y = -5 \end{cases}$$

Or

$$\begin{cases} -5x + y = 17 \\ 5x + 5y = 25 \end{cases}$$

Using the top one:

$$-6x = 12$$
$$x = -2$$

Plug back in to either equation (in the original system) to find the remaining variable(s) y=7.

Using the bottom system:

$$6y = 42$$
$$y = 7$$

Substitute to find x = -2.

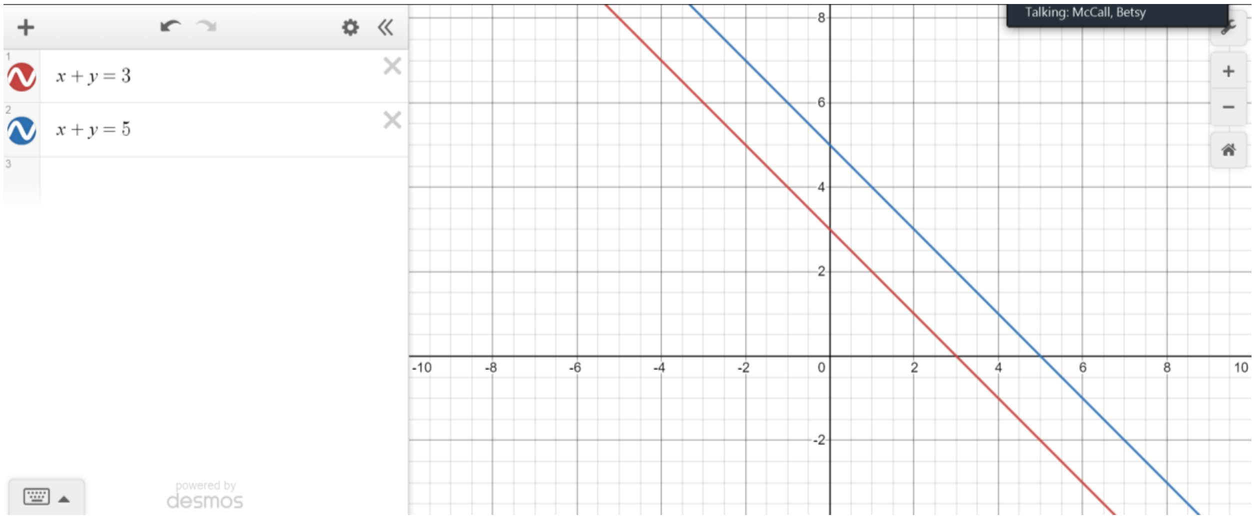
Check solution in second equation.

Types of systems and solutions:

Systems can consistent or inconsistent.
Consistent means there is at least one solution
Inconsistent means that there is no solution.

Inconsistent:
Has no point of intersection. No intersection means no solution.

Algebraically:
You get an equation where $x + 5 = x + 3$ or $0 = 2$
Unsolvable
Equations are parallel lines



Consistent can be one solution or infinite numbers of solutions: if the equations are multiples of each other, then the lines are the same, and any point that satisfies one equation, will also satisfy the other: infinite solutions.

Algebraically: look for $x = x$ or $0 = 0$.
Graphically they will appear as the same line.
When you solve for y, the equations will simplify to the same thing.

Solutions can also be classified as independent or dependent.

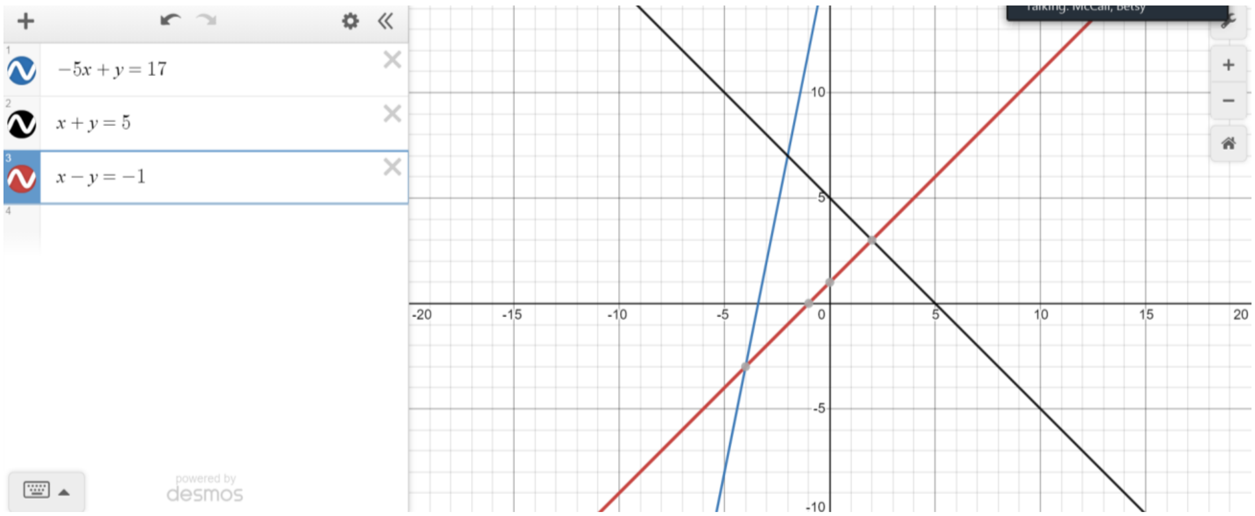
Independent solutions are those with a single pair of (x, y) that satisfies the system
Dependent solutions are those with infinite solutions.

The first system we solved was consistent because it had a solution. And independent because it had only one solution.

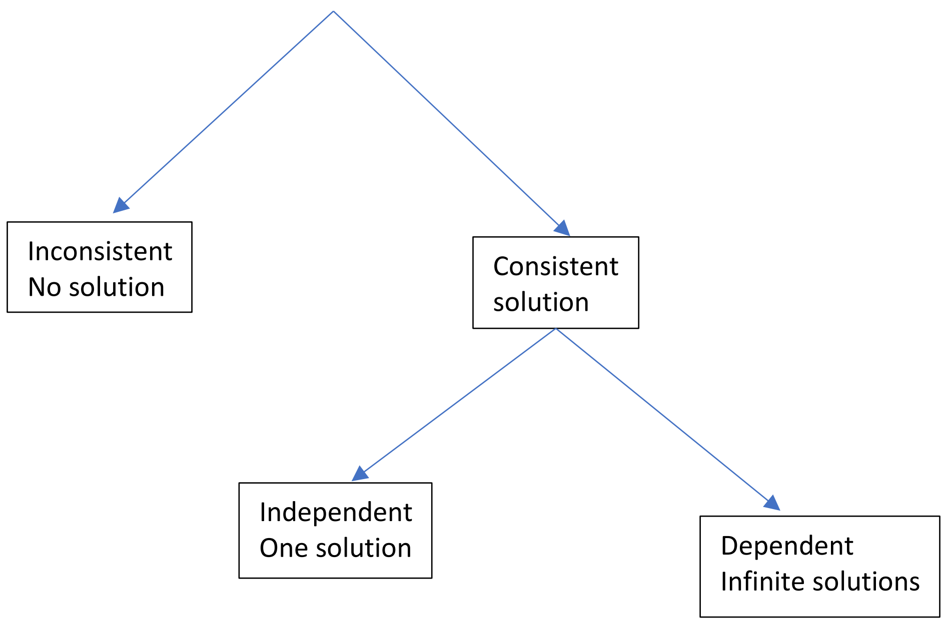
When we have two variables, we need a minimum of two equations in order to solve for a single point solution. (if we have fewer equations than variables it is impossible to solve for a single point solution, but having the minimum number of equations is not a guarantee of being able to solve for a single point solution.)

If we have too few equations to solve the system: this is called underdetermined: we don't have enough information. Fewer equations than we have variables.

If we have more equations than variables, then the system is called overdetermined: we have too much information.



Overdetermined may be unsolvable because the intersections of each pair of lines may not be the same. (or they may cross at the same place, and produce a solution—the more equations you have, the less likely you are to obtain a solution)



Example in three variables

$$\begin{cases} x + y + z = 4 \\ 2x - 4y - z = -1 \\ x - y \qquad = 2 \end{cases}$$

Using elimination:
Goal is to eliminate one of the variables and reduce the system to two equations in two variables.
Here: add the top two equations to eliminate z; z is already missing from the bottom equation

$$3x - 3y = 3$$

New reduced system

$$\begin{cases} 3x - 3y = 3 \\ x - y = 2 \end{cases}$$

Divide the top equation by 3 (alternatively, multiply the bottom equation by -3)

$$\begin{cases} x - y = 1 \\ x - y = 2 \end{cases}$$

$$\begin{cases} -x + y = -1 \\ x - y = 2 \end{cases}$$

Add

$$0 = 1$$

This system is inconsistent because 0 does not ever equal 1.

Convert our systems of equations into a matrix
Is essentially just a table of coefficients from the systems

$$\begin{cases} x + y + z = 4 \\ 2x - 4y - z = -1 \\ x - y = 2 \end{cases}$$

	x	y	z	constant
Equation 1	1	1	1	4
Equation 2	2	-4	-1	-1
Equation 3	1	-1	0	2

$$\begin{cases} -5x + y = 17 \\ x + y = 5 \end{cases}$$

$$\begin{bmatrix} -5 & 1 & 17 \\ 1 & 1 & 5 \end{bmatrix}$$

Sometimes the matrices will be written with a bar between the coefficients and the constants.

$$\left[\begin{array}{cc|c} -5 & 1 & 17 \\ 1 & 1 & 5 \end{array}\right]$$

The bar is just to emphasize that this is a system represented as an augmented matrix (augment is the column of constant)

How do we solve a system in matrix form: the technique is called Gaussian elimination, Gauss-Jordan elimination

Row operations:
Operation #1: Multiply a row by a constant $kR_i \to R_i$
Operation #2: Add rows (or multiples of rows) together $R_i + R_j \to R_i$
Operation #3: Interchange rows $R_i \leftrightarrow R_j$

Solve the two-variable system:

$$\left[\begin{array}{cc|c} -5 & 1 & 17 \\ 1 & 1 & 5 \end{array}\right]$$

Goal: Make the top left entry a_{11} (in the first row, first column) equal to 1
I'm going to switch the rows $R_1 \leftrightarrow R_2$

$$\left[\begin{array}{cc|c} 1 & 1 & 5 \\ -5 & 1 & 17 \end{array}\right]$$

Next: make all the entries below that value equal to zero by using elimination $5R_1 + R_2 \to R_2$

$$\begin{array}{ccc} 5 & 5 & 25 \\ -5 & 1 & 17 \end{array}$$

Add:

$$\begin{array}{ccc} 0 & 6 & 42 \end{array}$$

$$\left[\begin{array}{cc|c} 1 & 1 & 5 \\ 0 & 6 & 42 \end{array}\right]$$

Now, go to a_{22} (second row, second column): repeat: make this equal 1 (if possible) and eliminate any non-zero coefficients underneath it... and so on. When you get to the last row, then stop.

$$\frac{1}{6}R_2 \to R_2$$

$$\left[\begin{array}{cc|c} 1 & 1 & 5 \\ 0 & 1 & 7 \end{array}\right]$$

$$\begin{cases} x + y = 5 \\ \quad y = 7 \end{cases}$$

Back-solving: use the solution to the last variable to find the previous variable, and then successively all the remaining variables.
Here, we do indeed get x = -2.

This last form is called row echelon form. (echelon = stair in French) = ref

Let's look at the three-variable case:

$$\begin{cases} x + y + z = 4 \\ 2x - 4y - z = -1 \\ x - y \qquad = 2 \end{cases}$$

As a matrix:

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 4 \\ 2 & -4 & -1 & -1 \\ 1 & -1 & 0 & 2 \end{array}\right]$$

3×4 matrix (3 rows, 4 columns)
a_{11} to be 1 : already true
Then eliminate the coefficient below this 1: a_{21} make equal 0, and a_{31} make equal 0

$$-2R_1 + R_2 \rightarrow R_2$$

$$\begin{array}{cccc} -2 & -2 & -2 & -8 \\ 2 & -4 & -1 & -1 \end{array}$$

Add

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 4 \\ 0 & -6 & -3 & -9 \\ 1 & -1 & 0 & 2 \end{array}\right]$$

$$-R_1 + R_3 \rightarrow R_3$$

$$\begin{array}{cccc} -1 & -1 & -1 & -4 \\ 1 & -1 & 0 & 2 \end{array}$$

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 4 \\ 0 & -6 & -3 & -9 \\ 0 & -2 & -1 & -2 \end{array}\right]$$

Move to a_{22} make that a 1, and then use that 1 to eliminate all non-zeros below it.

$$-\frac{1}{6}R_2 \rightarrow R_2$$

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 4 \\ 0 & 1 & \frac{1}{2} & \frac{3}{2} \\ 0 & -2 & -1 & -2 \end{array}\right]$$

Make a_{32} equal to 0: $2R_2 + R_3 \rightarrow R_3$

$$\begin{array}{cccc} 0 & 2 & 1 & 3 \\ 0 & -2 & -1 & -2 \end{array}$$

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 4 \\ 0 & 1 & \frac{1}{2} & \frac{3}{2} \\ 0 & 0 & 0 & 1 \end{array}\right]$$

The 1’s positions we call the pivots because we use them to eliminate the coefficients below them
This form of the matrix is also in row echelon form

$$\begin{cases} x + y + z = 4 \\ y + \frac{1}{2}z = \frac{3}{2} \\ 0 = 1 \end{cases}$$

This system has no solution: because 0 does not equal 1

This textbook sometimes calls this triangular form.

For dependent system: in a row echelon form matrix, the bottom row is all zeros: 0=0 is a true statement

Reduced row echelon form (rref) -- to obtain this form by hand, you would continue Gaussian elimination. Once you got to the bottom row and last pivot, use the pivot to also make all the coefficients above those positions also equal to zero.

Curve fitting:
Suppose we have a set of points $(-1,3), (2,4), (5,2)$ and we want to fit a polynomial to the curve
We will a quadratic $ax^2 + bx + c = y$ has three unknowns.

$$a(-1)^2 + b(-1) + c = 3$$
$$a(2)^2 + b(2) + c = 4$$
$$a(5)^2 + b(5) + c = 2$$

$$\begin{cases} a - b + c = 3 \\ 4a + 2b + c = 4 \\ 25a + 5b + c = 2 \end{cases}$$

$$a = -\frac{1}{6}, b = \frac{1}{2}, c = \frac{11}{3}$$

$$y = -\frac{1}{6}x^2 + \frac{1}{2}x + \frac{11}{3}$$

Matrix operations

A matrix has dimensions $m \times n$

$$2 \times 3: \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}$$
$$2 \times 2: \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$$
$$4 \times 2: \begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \\ 7 & 8 \end{bmatrix}$$

If we want to add two matrices together, they must be the same size (same dimensions in rows and same in columns). Adding corresponding components

$$\begin{bmatrix}1 & 2\\3 & 4\end{bmatrix}+\begin{bmatrix}0 & -1\\3 & -4\end{bmatrix}=\begin{bmatrix}1 & 1\\6 & 0\end{bmatrix}$$

If they are not the same size, then the operation is not defined

Properties of addition:

$$A+B=B+A$$
$$A+(B+C)=(A+B)+C$$

Scalar multiplication:

Multiply every entry of the matrix by the same scalar

$$2\begin{bmatrix}1 & 2\\3 & 4\end{bmatrix}=\begin{bmatrix}2 & 4\\6 & 8\end{bmatrix}$$

$$k(A+B)=kA+kB$$
$$(j+k)A=jA+kA$$

Multiplying matrices by other matrices

Matrix multiplication works by multiplying the rows of the left matrix by the columns of the right matrix and adding them up.

The size of the columns of the left matrix must match the size of the rows of the right matrix

$$(m\times n)\cdot(n\times p)=(m\times p)$$

$$AB\ \neq BA$$

$$\begin{bmatrix}1 & 2\\3 & 4\end{bmatrix}\begin{bmatrix}0 & -1\\3 & -4\end{bmatrix}=\begin{bmatrix}(1)(0)+2(3) & (1)(-1)+(2)(-4)\\(3)(0)+(4)(3) & (3)(-1)+(4)(-4)\end{bmatrix}=\begin{bmatrix}6 & -9\\12 & -19\end{bmatrix}$$

$$\begin{bmatrix}0 & -1\\3 & -4\end{bmatrix}\begin{bmatrix}1 & 2\\3 & 4\end{bmatrix}=\begin{bmatrix}(0)(1)+(-1)(3) & (0)(2)+(-1)(4)\\(3)(1)+(-4)(3) & (3)(2)+(-4)(4)\end{bmatrix}=\begin{bmatrix}-3 & -4\\-9 & -10\end{bmatrix}$$

$$\begin{bmatrix}1 & 2\\3 & 4\\5 & 6\\7 & 8\end{bmatrix}\begin{bmatrix}1 & 2 & 3\\4 & 5 & 6\end{bmatrix}=\begin{bmatrix}(1)(1)+2(4) & (1)(2)+2(5) & (1)(3)+(2)(6)\\3(1)+4(4) & (3)(2)+4(5) & (3)(3)+4(6)\\(5)(1)+6(4) & (5)(2)+6(5) & (5)(3)+6(6)\\(7)(1)+(8)(4) & (7)(2)+(8)(5) & (7)(3)+(8)(6)\end{bmatrix}=\begin{bmatrix}9 & 12 & 15\\19 & 26 & 33\\29 & 40 & 51\\39 & 54 & 69\end{bmatrix}$$

If I have an equation of matrices, how do I solve it ?

$$Ax=b$$

Some matrices have inverses A^{-1} that act like dividing out the A matrix, but do it through multiplication

$$A^{-1}A=I$$
$$AA^{-1}=I$$

Identity matrix: it is a matrix that acts like a 1 in multiplication: 1’s on the diagonal and 0 everywhere else.

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$AI = A$$

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

$$A^{-1} = \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

$$ad - bc \neq 0$$

$$A^{-1} = \frac{1}{4 - 6} \begin{bmatrix} 4 & -2 \\ -3 & 1 \end{bmatrix} = \frac{1}{-2} \begin{bmatrix} 4 & -2 \\ -3 & 1 \end{bmatrix} = \begin{bmatrix} -2 & 1 \\ \frac{3}{2} & -\frac{1}{2} \end{bmatrix}$$

$$Ax = b$$
$$A^{-1}Ax = A^{-1}b$$
$$x = A^{-1}b$$

$$\left[\begin{array}{cc|c} -5 & 1 & 17 \\ 1 & 1 & 5 \end{array}\right]$$

$$A = \begin{bmatrix} -5 & 1 \\ 1 & 1 \end{bmatrix}$$

$$\vec{x} = \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$b = \begin{bmatrix} 17 \\ 5 \end{bmatrix}$$

$$\begin{bmatrix} -5 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 17 \\ 5 \end{bmatrix}$$

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -5 & 1 \\ 1 & 1 \end{bmatrix}^{-1} \begin{bmatrix} 17 \\ 5 \end{bmatrix}$$