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2.5 Formal (epsilon-delta) Definition of a limit

$f(x)$ is defined for values of x near a, but perhaps not at a itself (this definition allows for a hole—removable discontinuity). Then $\lim_{x \to a} f(x) = L$, if:
For every $\varepsilon > 0$, there exists a $\delta > 0$ such that if $0 < |x - a| < \delta$, then $|f(x) - L| < \varepsilon$.

$f(x) = 2x - 1, a = 2, L = 3$

What to find a relationship between ε and δ, so that if we know one, then we know the other (if we know how close we are in x, we know how close we are in y).

If $|x - 2| < \delta$, then:

$$|2x - 1 - 3| < \varepsilon$$
$$|2x - 4| < \varepsilon$$
$$2|x - 2| < \varepsilon$$
$$2\delta < \varepsilon$$
$$\delta < \frac{\varepsilon}{2}$$

If we know epsilon, we know how close to the limit we need to be (say 0.1), then we can use this formula to determine how close to the value a we have to be in x to get an estimate (0.1/2=0.05).

Prove that the limit of $f(x)$ is equal to L as x approaches a using the $\varepsilon - \delta$ definition of the limit.

Example.
Prove that $\lim_{x \to -1} 5x + 8 = 3$ using the $\varepsilon - \delta$ definition of the limit.

If $|x - (-1)| < \delta$, then $|5x + 8 - 3| < \varepsilon$.

$$|x + 1| < \delta$$
$$|5x + 5| < \varepsilon$$
$$5|x + 1| < \varepsilon$$
$$5\delta < 5|x + 1| < \varepsilon$$
$$5\delta < \varepsilon$$
$$\delta < \frac{\varepsilon}{5}$$

Therefore, the $\lim_{x \to -1} 5x + 8$ is equal to 3.

Example.
Prove that $\lim_{x \to 6} 4 - \frac{x}{2} = 1$ using the epsilon-delta definition of the limit.

If $|x - 6| < \delta$, then $\left|4 - \frac{x}{2} - 1\right| < \varepsilon$.

$$|x - 6| < \delta$$
$$\left|3 - \frac{x}{2}\right| < \varepsilon$$

$$\left|\frac{6}{2}-\frac{x}{2}\right| = \left|\frac{6-x}{2}\right| = \left|\frac{x-6}{-2}\right| < \varepsilon$$

$$\left|\frac{x-6}{-2}\right| = \frac{|x-6|}{|-2|} = \frac{|x-6|}{2} = \frac{1}{2}|x-6| < \varepsilon$$

$$\frac{1}{2}\delta < \frac{1}{2}|x-6| < \varepsilon$$

$$\frac{1}{2}\delta < \varepsilon$$

$$\delta < 2\varepsilon$$

Therefore:
For every $\varepsilon > 0$, there exists a $\delta > 0$ such that if $0 < |x-a| < \delta$, then $|f(x) - L| < \varepsilon$.

Given $\varepsilon > 0$, and $\delta < 2\varepsilon$, then $|x-6| < \delta$ implies: $|x-6| < 2\varepsilon$, which implies $\frac{1}{2}|x-6| < \varepsilon$, which implies $\left|\frac{x}{2}-3\right| = \left|3-\frac{x}{2}\right| < \varepsilon$, which implies $\left|\left(4-\frac{x}{2}\right)-1\right| < \varepsilon$. Therefore: $\lim_{x\to 6} 3-\frac{x}{2} = 1$.

“backward solving”. Technically, the technique starts with the “then” and works backwards to the “if” statement to reverse engineer the relationships needed. Then the proof itself, starts with the “if”, and uses the relationships obtained to get to the then statement.

Look at an example using a quadratic function to see what happens when the functions are nonlinear.

$$\lim_{x\to 2}(x^2+2x) = 8$$

Prove, using the epsilon-delta definition of the limit.

Example.

$$|x-2| < \delta$$
$$|x^2+2x-8| < \varepsilon$$
$$|(x-2)(x+4)| < \varepsilon$$
$$|x-2||x+4| < \varepsilon$$

Need to replace, $|x+4|$ with a constant. I need to find an estimate for this expression that near $x = 2$, will be a maximum value. What is the largest value this expression can have if you are within 1 unit of $x = 2$?

Test: $x = 1, x = 2, x = 3$… pick the biggest. $|1+4| = 5, |2+4| = 6, |3+4| = 7$. For values “near” $x = 2$, $|x+4| \leq 7$.

$$|x-2||x+4| < 7|x-2| < \varepsilon$$
$$7\delta < 7|x-2| < \varepsilon$$

$$7\delta < \varepsilon$$
$$\delta < \frac{\varepsilon}{7}$$

For infinite limits, you select a number M and show that the difference between the function and M doesn't shrink, that there is always some values of x for which f(x) is bigger than M.

One sided limits basically work the same way as two-sided limits, but you only need to worry about one side: so that absolute values aren't necessary (they don't hurt to have them).