2/1/2022

3.1 Definition of the Derivative
3.2 Derivative as a function
3.3 Derivative Rules (short-cuts)

Secant line slope:

$$m_{sec} = \frac{f(b) - f(a)}{b - a} = \frac{f(x) - f(a)}{x - a} = \frac{f(x+h) - f(x)}{h}$$

Slope of tangent line:

$$m_{tan} = \lim_{x \to a} \frac{f(x) - f(a)}{x - a} = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

The derivative is a function: $f'(x)$ if the function is $f(x)$. ($f'(x)$ is read "f-prime of x") When you evaluate the function at a specific point (a), then the value is the slope of the tangent line at $x = a$.

The first definition is evaluated at a already. The second one produces a function that is evaluated at $x = a$.

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

$$f'(a) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}$$

What is the derivative (slope of the tangent line) of the function $f(x) = 3x + 2$? Using the definition.

$$\lim_{h \to 0} \frac{3(x+h) + 2 - (3x+2)}{h} = \lim_{h \to 0} \frac{3x + 3h + 2 - 3x - 2}{h} = \lim_{h \to 0} \frac{3h}{h} = \lim_{h \to 0} 3 = 3$$

$$f'(x) = 3$$

Consider an extension here where the line is horizontal. The slope of the line is zero, and so the derivative is zero. The derivative is the rate of change, so for a line, it's just equal to the constant slope.

Find the derivative of the function $g(x) = x^2 + x$
(aside: $f(x+h) \neq f(x) + h$ and $(x+h)^2 \neq x^2 + h^2$, $(x+h)^2 = x^2 + 2xh + h^2$)

$$g'(x) = \lim_{h \to 0} \frac{(x+h)^2 + (x+h) - (x^2 + x)}{h} = \lim_{h \to 0} \frac{x^2 + 2xh + h^2 + x + h - x^2 - x}{h} =$$

$$\lim_{h \to 0} \frac{2xh + h^2 + h}{h} = \lim_{h \to 0} \frac{h(2x + h + 1)}{h} = \lim_{h \to 0} (2x + h + 1) = 2x + 1$$

This function will give me the slope of the tangent line at any point on the curve $g(x) = x^2 + x$.

What is slope of the tangent line at $x = 2$?

$$g'(x) = 2x + 1, g'(2) = 5$$

What is the equation of the tangent line? Find a point on the curve. $g(2) = 6$. Use the point-slope form.

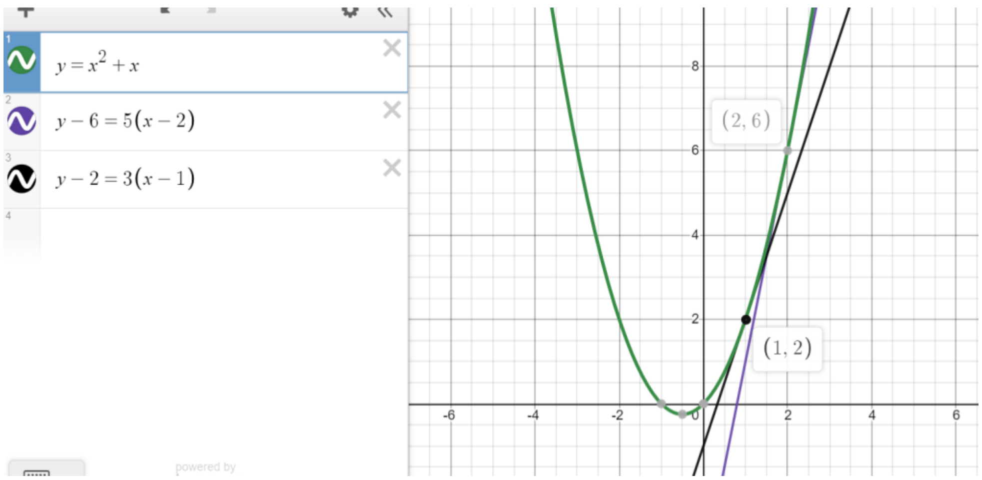
$$y - 6 = 5(x - 2)$$

What is the slope of the tangent line at $x = 1$? What is the instantaneous rate of change at $x = 1$?

$$g'(1) = 3$$

What is the equation of the tangent line? $g(1) = 1$.

$$y - 2 = 3(x - 1)$$



$$(x + h)^2 = (x + h)(x + h) = x^2 + 2xh + h^2$$

$$(x + h)^3 = (x + h)(x + h)(x + h) = (x + h)(x^2 + 2xh + h^2) = x^3 + 3x^2h + 3xh^2 + h^3$$

$$(x + h)^4 = (x + h)^2(x + h)^2 = (x^2 + 2xh + h^2)(x^2 + 2xh + h^2) =$$

$$x^4 + 4x^3h + 6x^2h^2 + 4xh^3 + h^4$$

$$k(x) = 3x^2 - 5x + 6$$

Find the derivative of $k(x)$.

$$\lim_{h \to 0} \frac{3(x + h)^2 - 5(x + h) + 6 - (3x^2 - 5x + 6)}{h} =$$

$$\lim_{h \to 0} \frac{3(x^2 + 2xh + h^2) - 5(x + h) + 6 - (3x^2 - 5x + 6)}{h} =$$

$$\lim_{h \to 0} \frac{3x^2 + 6xh + 3h^2 - 5x - 5h + 6 - 3x^2 + 5x - 6}{h} =$$

$$\lim_{h \to 0} \frac{6xh + 3h^2 - 5h}{h} = \lim_{h \to 0} \frac{h(6x + 3h - 5)}{h} = \lim_{h \to 0} 6x + 3h - 5 = 6x - 5$$

What other kinds of functions? Rational functions, radical functions.

$$f(x) = \frac{1}{x+1}$$

Find the derivative of $f(x)$.

$$\lim_{h\to 0} \frac{\frac{1}{x+h+1} - \frac{1}{x+1}}{h} = \lim_{h\to 0} \frac{1}{h}\left[\frac{1}{x+h+1} - \frac{1}{x+1}\right] =$$

$$\lim_{h\to 0} \frac{1}{h}\left[\frac{1}{x+h+1}\cdot\frac{x+1}{x+1} - \frac{1}{x+1}\cdot\frac{x+h+1}{x+h+1}\right] =$$

$$\lim_{h\to 0} \frac{1}{h}\left[\frac{x+1}{(x+h+1)(x+1)} - \frac{x+h+1}{(x+1)(x+h+1)}\right] =$$

$$\lim_{h\to 0} \frac{1}{h}\left[\frac{x+1-(x+h+1)}{(x+h+1)(x+1)}\right] = \lim_{h\to 0} \frac{1}{h}\left[\frac{x+1-x-h-1}{(x+h+1)(x+1)}\right] =$$

$$\lim_{h\to 0} \frac{1}{h}\left[\frac{-h}{(x+h+1)(x+1)}\right] = \lim_{h\to 0}\left[\frac{-1}{(x+h+1)(x+1)}\right] = \frac{-1}{(x+1)(x+1)} = -\frac{1}{(x+1)^2}$$

Substitute into the difference quotient. Pull out 1/h from denominator. Then combine the two fractions into a single fraction. Simplify the numerator. Cancel the h. Substitute the limit value h=0.

$$f(x) = \sqrt{x-4}$$

Find the derivative of $f(x)$, using the limit definition.

$$\lim_{h\to 0} \frac{\sqrt{x+h-4} - \sqrt{x-4}}{h} = \lim_{h\to 0} \frac{\sqrt{x+h-4} - \sqrt{x-4}}{h} \cdot \frac{\sqrt{x+h-4} + \sqrt{x-4}}{\sqrt{x+h-4} + \sqrt{x-4}} =$$

$$\lim_{h\to 0} \frac{\left(\sqrt{x+h-4}\right)^2 - \left(\sqrt{x-4}\right)^2}{h\left(\sqrt{x+h-4} + \sqrt{x-4}\right)} = \lim_{h\to 0} \frac{x+h-4-(x-4)}{h\left(\sqrt{x+h-4} + \sqrt{x-4}\right)} =$$

$$\lim_{h\to 0} \frac{x+h-4-x+4}{h\left(\sqrt{x+h-4} + \sqrt{x-4}\right)} = \lim_{h\to 0} \frac{h}{h\left(\sqrt{x+h-4} + \sqrt{x-4}\right)} =$$

$$\lim_{h\to 0} \frac{1}{\left(\sqrt{x+h-4} + \sqrt{x-4}\right)} = \frac{1}{\left(\sqrt{x-4} + \sqrt{x-4}\right)} = \frac{1}{2\sqrt{x-4}}$$

Substitute into the difference quotient. Multiply by the conjugate/conjugate (wherever the radical is). Do not multiply the part of the fraction with the h. Multiply where the radical expression and the conjugate are multiplying each other. You should be able to do algebra to get to a point where you can cancel the h in the denominator and then substitute the limit value.

Derivative:
Rate (of change) -- be careful of average rate of change (traditional slope), vs. instantaneous rate of change (derivative) – average rate of change needs 2 points specified; instantaneous rate of change at single point.

Tangent line

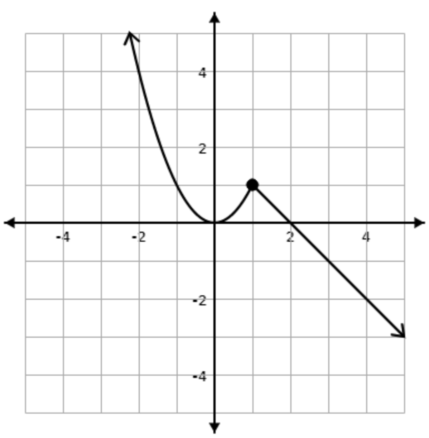
Velocity – position function's derivative is the velocity function. (how fast is it going?)
Acceleration – the derivative of the velocity (how fast is the speed changing?)

Marginal – "marginal cost" is the derivative of the cost

Differentiability – if a function produces a continuous derivative, then it is called differentiable.

Polynomials are smooth (no sharp turns or jumps) and are always differentiable.
Piecewise functions, to be differentiable, need to have the pieces come together at the same point (they need to be continuous), AND the derivatives of those pieces must also have the same limit.

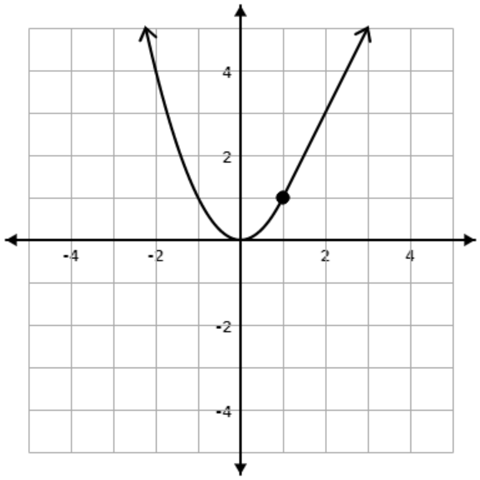
Not differentiable:



The derivative of the first piece at x=1 is 2. And the derivative of the second piece at x=1, is -1.

$$f(x) = \begin{cases} x^2, x \le 1 \\ 2 - x, x > 1 \end{cases}$$

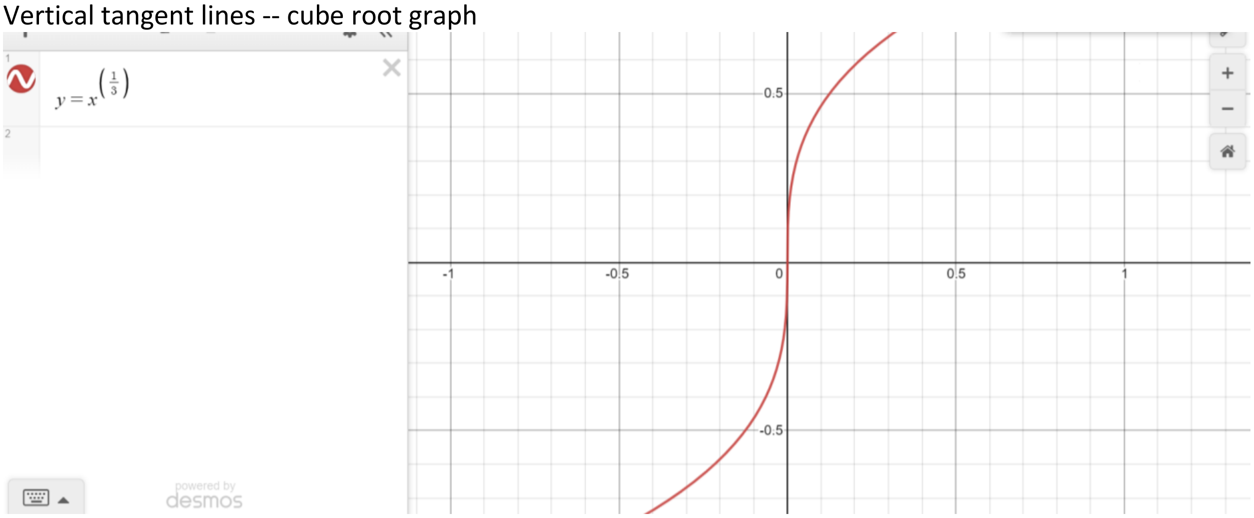
Differentiable:



$$f(x) = \begin{cases} x^2, x \leq 1 \\ 2x - 1, x > 1 \end{cases}$$

In the second case, the function is continuous (the limits on the left- and right-sides are equal), and the derivatives of the pieces also are equal at x=1.

Things to look for to determine if a function is not differentiable:
Jumps in the graph (discontinuities) – vertical asymptotes, piecewise functions, graphs with holes
Cusps (v-shaped parts of the graph) – absolute value
Vertical tangent lines -- cube root graph



If a function is differentiable (continuous derivative), then the function is also continuous.

Notation for first derivative:

$$f'(x), D(f), D_x(f), \dot{f}(t), \frac{df}{dx}$$

(first one is Newtonian, the last one is Leibniz notation).

Second derivative:

$$f''(x), \ddot{f}(t), \frac{d^2 f}{dx^2}$$

Third derivative:

$$f'''(x), \dddot{f}(t), \frac{d^3 f}{dx^3}$$

Fourth derivative:

$$f^{IV}(x), f^{(4)}(x), \frac{d^4 f}{dx^4}$$

Nth derivative:

$$f^{(n)}(x), \frac{d^n f}{dx^n}$$

$f^4(x)$ is the 4th power of the function $f(x)$, = $[f(x)]^4$
$f^{(4)}(x)$ is the fourth derivative of the function $f(x)$. is equivalent to $\frac{d}{dx}[f'''(x)]$
The operator for "take the derivative" is $\frac{d}{dx}$

Pick up with 3.3 next time.