2/15/2022

3.8 Implicit Differentiation
3.9 Exponential and Logarithm Derivatives

Implicit Differentiation
Functions can be expressed implicitly or explicitly.
Explicit function are expressions that use function notation, or can be written in function notation: you can solve for y explicitly. $y = f(x)$
Implicitly defined functions can't be solved algebraically for y. (we described these as relations)

$$x^2 + y^2 = 4$$

When you find the derivative of an implicit function $\frac{dy}{dx}$, the x-variable is going to be treated "normally", but when we find the derivative of the function variable (y), then we will apply the chain rule. We are treating y as a function of x, but we just don't know what that function is. For example, the $\frac{d}{dx}(y^2) = 2y \cdot \frac{dy}{dx}$ or $2y\, y'$.

If there is a term that involves both x and y, then since we are treating y as a function of x, which is multiplied by another x, we will have to do the product. (Likewise if x and y are dividing each other: use quotient rule).

Find the derivative of $x^2 + y^2 = 4$ implicitly.

$$\frac{d}{dx}(x^2 + y^2) = \frac{d}{dx}(4)$$

$$\frac{d}{dx}(x^2) + \frac{d}{dx}(y^2) = \frac{d}{dx}(4)$$

$$2x + 2y\, y' = 0$$

Solve for y' to get $\frac{dy}{dx}$.

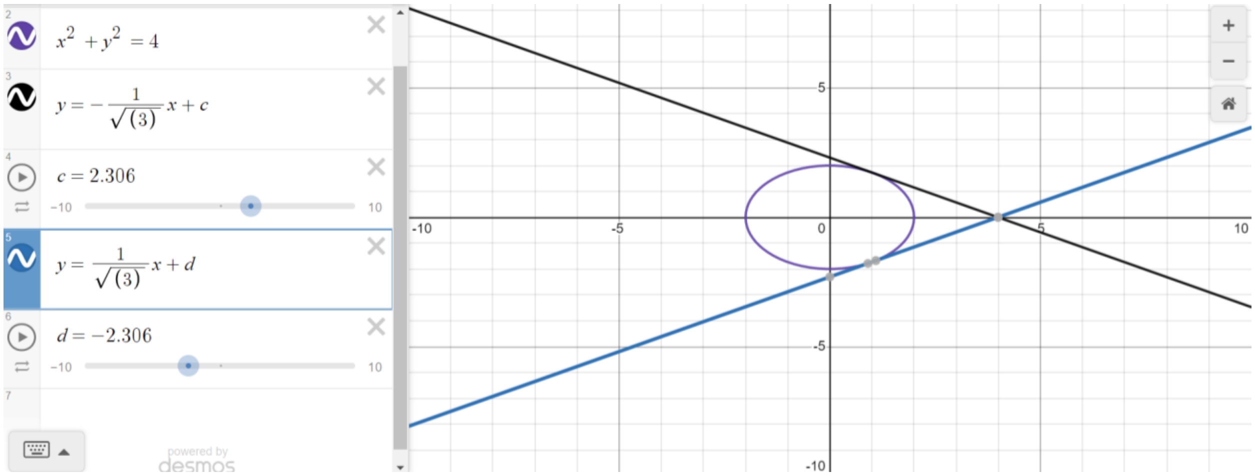
$$2y\, y' = -2x$$

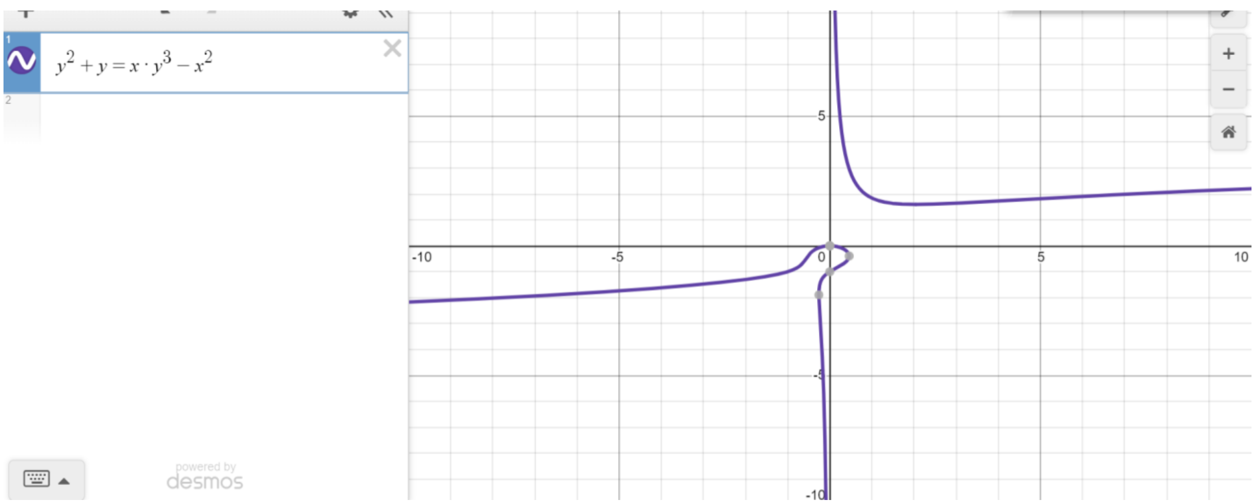
$$y' = -\frac{2x}{2y}$$

$$\frac{dy}{dx} = -\frac{x}{y}$$

If I want to find the slope of the tangent line at a specific point, then I need both x and y values. $(1, \sqrt{3})$

The slope of the tangent at this point is $\frac{dy}{dx} = -\frac{1}{\sqrt{3}}$.





$$y^2 + y = xy^3 - x^2$$

Find $\frac{dy}{dx}$ implicitly.

$$2y\,y' + y' = {\color{red}y^3 + 3xy^2\,y'} - 2x$$

$f(x) = x$	$g(x) = y^3$
$f'(x) = 1$	$g'(x) = 3y^2\,y'$

Collect all y' terms on one side of the equation, and all terms without y' on the other.

$$2y\,y' + y' - 3xy^2\,y' = y^3 - 2x$$

$$y'(2y + 1 - 3xy^2) = y^3 - 2x$$

$$y' = \frac{dy}{dx} = \frac{y^3 - 2x}{2y + 1 - 3xy^2}$$

Find the slope of the tangent line at (0,0).

$$\frac{dy}{dx}(0,0) = \frac{0}{1} = 0$$

Find the slope of the tangent line at $(-1,-1)$.

$$\frac{dy}{dx}(-1,-1) = \frac{-1+2}{-2+1+3} = \frac{1}{2}$$

$$\cos(xy) + y\tan x + 1 = y^2 + x$$

$$-\sin(xy) \cdot \frac{d}{dx}(xy) + \frac{d}{dx}(y\tan x) + 0 = 2y\,y' + 1$$

$f(x) = x$	$g(x) = y$
$f'(x) = 1$	$g'(x) = y'$

$p(x) = y$	$q(x) = \tan x$
$p'(x) = y'$	$q'(x) = \sec^2 x$

$$-\sin(xy) \cdot (y + xy') + (y'\tan x + y\sec^2 x) = 2y\,y' + 1$$

$$-y\sin xy - xy'\sin xy + y'\tan x + y\,sec^2 x = 2y\,y' + 1$$

$$-xy'\sin xy + y'\tan x - 2y\,y' = 1 + y\sin xy - y\,sec^2 x$$

$$y'(-x\sin xy + \tan x - 2y\,) = 1 + y\sin xy - y\,sec^2 x$$

$$\frac{dy}{dx} = y' = \frac{1 + y\sin xy - y\,sec^2 x}{-x\sin xy + \tan x - 2y}$$

$$\arctan(y^3) = xy + \sec x$$

Find the implicit derivative $\frac{dy}{dx}$.

$$\frac{1}{1+(y^3)^2} \cdot \frac{d}{dx}(y^3) = y + xy' + \sec x\tan x$$

$$\frac{1}{1+y^6} \cdot 3y^2 y' = y + xy' + \sec x\tan x$$

$$\frac{3y^2}{1+y^6} \cdot y' - xy' = y + \sec x\tan x$$

$$y'\left(\frac{3y^2}{1+y^6} - x\right) = y + \sec x\tan x$$

$$\frac{dy}{dx} = y' = \frac{y + \sec x \tan x}{\frac{3y^2}{1 + y^6} - x}$$

Exponential and Logarithm Derivatives

The exponential function, the natural exponential $y = e^x$

$$\frac{d}{dx}(e^x) = e^x$$

For the natural exponential function, the slope of the tangent line is equal to the y-value at that point.

If the base of the exponential function is not e, then the derivative rule has a constant multiplier to the rule.
a is a positive real number not equal to 1.

$$\frac{d}{dx}(a^x) = (\ln a)a^x$$

The rule for the derivative of the natural log function $y = \ln x = \log_e x$

$$\frac{d}{dx}(\ln x) = \frac{1}{x}$$

$$e^y = x$$
$$e^y y' = 1$$
$$y' = \frac{1}{e^y} = \frac{1}{x}$$

The rule for the derivative of the log function base-a? $y = \log_a x$

$$\frac{d}{dx}(\log_a x) = \frac{1}{\ln a} \cdot \frac{1}{x}$$

$$a^y = x$$
$$(\ln a)a^y y' = 1$$

$$y' = \frac{1}{(\ln a)a^y} = \frac{1}{(\ln a)x}$$

Examples.

Find the derivative of $f(x) = 4e^x + \ln(2x) - \cot(x) = 4e^x + \ln(2) + \ln(x) - \cot(x)$

$$f'(x) = 4e^x + \frac{1}{2x}(2) + \csc^2 x = 4e^x + \frac{1}{x} + \csc^2 x$$

$$g(x) = xe^x + \ln(e^x + 6x)$$

$$g'(x) = 1 \cdot e^x + x \cdot e^x + \frac{1}{e^x + 6x} \cdot (e^x + 6)$$

$$g'(x) = e^x + xe^x + \frac{e^x + 6}{e^x + 6x}$$

$$h(x) = e^{x^2} + \left(\frac{1}{2}\right)^x$$

$$h'(x) = e^{x^2}(2x) + \left(\ln\frac{1}{2}\right)\left(\frac{1}{2}\right)^x = 2xe^{x^2} + \left(\ln\frac{1}{2}\right)\left(\frac{1}{2}\right)^x$$

$$F(x) = \log_8(x^3 + 1) = \frac{\ln(x^3 + 1)}{\ln 8} = \frac{1}{\ln 8}\ln(x^3 + 1)$$

$$F'(x) = \frac{1}{\ln 8} \cdot \frac{1}{x^3 + 1} \cdot 3x^2 = \frac{3x^2}{(\ln 8)(x^3 + 1)}$$

$$a^x = \left(e^{\ln a}\right)^x = e^{x \ln a}$$
$$\frac{d}{dx}\left(e^{x \ln a}\right) = e^{x \ln a}(\ln a) = a^x(\ln a)$$

Reminder about log rules:

$$\ln(MN) = \log M + \log N$$
$$\ln\left(\frac{M}{N}\right) = \log M - \log N$$
$$\ln(M^r) = r \log M$$

If you have a particularly complex log function, do the algebra first.

$$f(x) = \ln\left(\frac{x^2 \sin x}{2x + 1}\right) = \ln(x^2 \sin x) - \ln(2x + 1) = \ln(x^2) + \ln(\sin x) - \ln(2x + 1) =$$

$$2 \ln x + \ln(\sin x) - \ln(2x + 1)$$

$$f'(x) = \frac{2}{x} + \frac{1}{\sin x}(\cos x) - \frac{1}{2x + 1}(2) = \frac{2}{x} + \cot(x) - \frac{2}{2x + 1}$$

The next thing is using log properties to find derivatives of some functions for which we don't have a simple rule, like $f(x) = x^x$.

We will pick this topic up next time.
We will cover the derivatives of hyperbolic trig functions.
Review section 1.5 – covers the definition and algebra of hyperbolic trig functions.