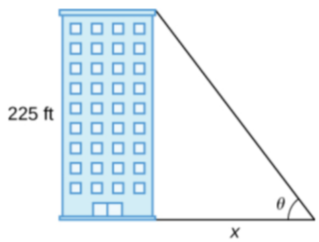
2/22/22

Happy TWOsday!

4.1 Related Rates
4.2 Linear Approximations and Differentials

● Question 16 ☑ 0/10 pts ↺ 100 ⇄ 99 ⓘ Details



A building that is 225 feet tall casts a shadow of various lengths as the day goes by. An angle of elevation θ is formed by lines from the top and bottom of the building to the tip of the shadow, as seen in the figure above.

Find the rate of change of the angle of elevation $\frac{d\theta}{dx}$ when $x = 121$.

$\theta'(121) =$ ______ radians per foot. Round to five decimal places.

$$\tan\theta = \frac{225}{x}$$

$$\arctan\left(\frac{225}{x}\right) = \tan^{-1}\left(\frac{225}{x}\right) = \theta$$

$$\frac{d\theta}{dx} = \frac{1}{1+\left(\frac{225}{x}\right)^2}\cdot\left(-\frac{225}{x^2}\right)$$

$$u = 225x^{-1}$$
$$u' = 225(-1x^{-2})$$

$$\frac{d\theta}{dx} = \frac{1}{1+u^2}\cdot u'$$

Related Rates

In a parametric function, you have a pair of x and y values (that are related to each other), they form one curve, but each variable is actually dependent on time.

Example. Suppose you drop a pebble in a pond. This sets up ripples on the water. Those ripples are circular. The circumference of the circle is always $C = 2\pi r$, but since the ripples are expanding outward from where the pebble fell, the radius r is changing with time $r(t)$. That means the circumference is also changing with time.

If the radius and circumference are changing (they are not constant), then the rates of change are also changing with time, and the rates of change are also related.

Area is $A = \pi r^2$, the rate of change will depend on the rate of change of r, but also r itself.

Starting with the circumference.

$$C = 2\pi r$$
$$C(t) = 2\pi r(t)$$

The assumptions we are making are that all the variables in the equation are themselves functions of time.
That means when we take the derivative **with respect to time** we need to take the derivative of all variables implicitly.

$$\frac{dC}{dt} = 2\pi \frac{dr}{dt}$$

Typically, the problem will then give you some information about one of the rates (and any remaining variables), and then you have to solve for the other.

Suppose $\frac{dr}{dt} = 1\frac{cm}{s}, \frac{dC}{dt} = 2\pi\left(1\frac{cm}{s}\right) \approx 6.28\frac{cm}{s}$

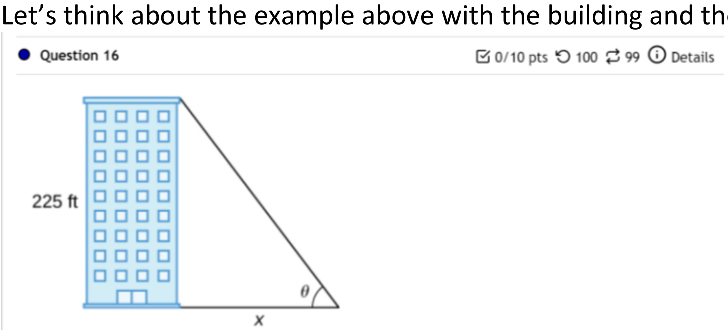
$$A(t) = \pi r^2(t) = \pi[r(t)]^2$$

$$\frac{dA}{dt} = \pi(2r)\left(\frac{dr}{dt}\right) = 2\pi r\frac{dr}{dt}$$

Suppose $\frac{dr}{dt} = 1\frac{cm}{s}, r = 6\ s$
Then $\frac{dA}{dt} = 2\pi(6)\left(1\frac{cm}{s}\right) = 12\pi\frac{cm}{s} \approx 37.7\frac{cm}{s}$

Step 1: set up the (usually geometric) equation that relates the variables in the problem.
Step 2: find the related rate equation (take the derivative with respect to time)
Step 3: you may also need a relationship between other variables in the problem (this is to reduce the number of variables in the problem).
Step 4: plug in whatever information they've given you about one rate, and the other variables

Let's think about the example above with the building and the changing angle.



The building has a fixed height, but the there is a person x feet away from the building walking backward at a rate of $\frac{dx}{dt} = 1\frac{ft}{s}$.
What is the rate of change of the angle (in radians) when the person is 150 feet away.

$$\tan\theta = \frac{225}{x}$$

$$\sec^2\theta \cdot \frac{d\theta}{dt} = -\frac{225}{x^2}\cdot\frac{dx}{dt}$$

Hypotenuse is $\sqrt{225^2 + x^2}$ using the Pythagorean theorem.

$$\sec\theta = \frac{\sqrt{225^2 + x^2}}{x}$$

$$\left(\frac{\sqrt{225^2 + x^2}}{x}\right)^2 \cdot \frac{d\theta}{dt} = -\frac{225}{x^2}\cdot\frac{dx}{dt}$$

$$(225^2 + x^2)\frac{d\theta}{dt} = -225\frac{dx}{dt}$$

$$\frac{d\theta}{dt} = -\frac{225}{225^2 + x^2}\frac{dx}{dt}$$

$$\frac{d\theta}{dt} = -\frac{225}{225^2 + 150^2}\left(1\frac{ft}{s}\right) \approx -0.0030769\ldots\frac{radians}{s}$$

In units of radians per second (feet units cancel).

A conveyor belt is dropping refined sand onto a conical pile with a circular base that has a base whose radius is twice as large as its height. The volume of the pile is changing at a rate of $1000\frac{ft^3}{s}$. Find the rate at which the height of the pile is changing if the height is currently 50 feet.

$$V = \frac{1}{3}\pi r^2 h$$

$$r = 2h$$

$$V = \frac{1}{3}\pi(2h)^2 h = \frac{2}{3}\pi h^3$$

$$\frac{dV}{dt} = \frac{2}{3}\pi(3h^2)\frac{dh}{dt} = 2\pi h^2\frac{dh}{dt}$$

$$\frac{1}{2\pi h^2}\frac{dV}{dt} = \frac{dh}{dt}$$

$$\frac{1}{2\pi(50^2)}\left(1000\frac{ft^3}{s}\right) = \frac{dh}{dt} \approx 0.06366\ldots\frac{ft}{s}$$