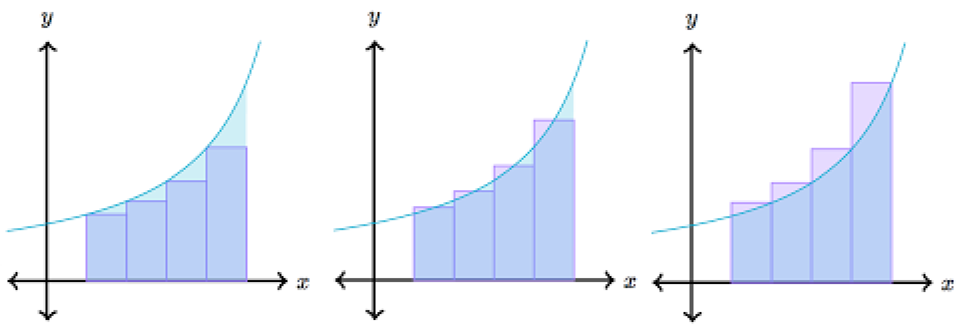
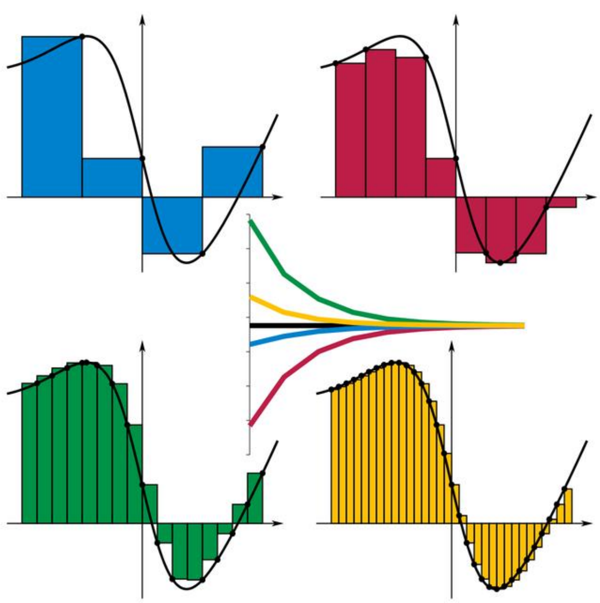
3/29/2022

5.1 Approximating Area
5.2 Definite Integral

Approximating area
Estimate the area under a generic curve by using rectangles (other shapes are possible). Divide up the region on the x-axis into a fixed number of equally sized sections. Then choose a value of the function inside that section to be the height of the rectangle for our estimate. The width is the width of section. Typically, we will pick the right or left endpoints. It turns out that in the limit as n gets bigger, all estimates end up at the same value. Some methods will choose the largest of the function or smallest on that section (for getting an upper or lower bound), some methods will choose the midpoint (similar to the trapezoidal rule—see Calc 2).





Summation notation

$$\sum_{i=1}^{n} i\,, \sum_{i=1}^{n} f(x_i)$$

$$\sum_{i=1}^{5} i = 1 + 2 + 3 + 4 + 5$$

$$\sum_{i=1}^{4} f(x_i) = f(x_1) + f(x_2) + f(x_3) + f(x_4)$$

Sums of powers of i

$$\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$$

$$\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}$$

$$\sum_{i=1}^{n} i^3 = \frac{n^2(n+1)^2}{4}$$

$$\sum_{i=1}^{n} c = cn$$

Example.

$$\sum_{k=1}^{20} (2k+1) = \sum_{k=1}^{20} 2k + \sum_{k=1}^{20} 1 = 2\sum_{k=1}^{20} k + \sum_{k=1}^{20} 1 = 2\left(\frac{20(20+1)}{2}\right) + (1)(20) = 420 + 20 = 440$$

$$\sum_{i=10}^{20} i = \sum_{i=1}^{20} i - \sum_{i=1}^{9} i$$

How do we use these to approximate the area?

The area under a curve is going to be estimated by a fixed number of rectangles.
The width of the rectangle Δx_i (but usually Δx)
The height of the rectangles is $f(x_i)$. These x_i's are values inside each segment of the graph (after dividing up the x-axis.
To estimate the area, we multiply the width by the height, and then add up the rectangles.
n is the number of rectangles we are using in the estimate

$$A \approx \sum_{i=1}^{n} f(x_i)\Delta x$$

When we divide up the interval over which we are calculating the area, the list of x-values that define the way we've divided it up is called a partition.

Suppose my interval is (I'm calculating the area under a curve between) [1,4], and I'm using 5 rectangles.

$$n = 4$$

Our formula for Δx since we are using intervals of constant width is $\Delta x = \frac{b-a}{n}$. b is the second endpoint on the interval, and a is the first $[a, b]$.

In this case, $\Delta x = \frac{4-1}{5} = \frac{3}{5} = 0.6$

Partition: $\{a = x_0 = 1, x_1 = 1.6, x_2 = 2.2, x_3 = 2.8, x_4 = 3.4, b = x_5 = 4\}$
Subintervals: [1,1.6], [1.6,2.2], [2.2,2.8], [2.8,3.4], [3.4,4]
Formula for the sub-intervals $[a + (i-1)\Delta x, a + i\Delta x]$

Left-hand rule says use the x-value on the left-end of each subinterval: $\{x_0, x_1, x_2, \dots, x_4\}$
Right-hand rule says use the x-value on the right-end of each subinterval: $\{x_1, x_2, x_3, \dots, x_5\}$

Midpoint rule says to use the midpoint of each subinterval: $\left\{\frac{x_0+x_1}{2}, \frac{x_1+x_2}{2}, \dots \frac{x_4+x_5}{2}\right\}$

For the maximum/upper bound estimate: you have to determine where in each subinterval the function was largest. Absolute maximum problem (you have to do it one time for every subinterval). If the function is monotonic, the maximum corresponds to the right-hand rule if increasing, and the left-hand rule if decreasing (on each subinterval).

Likewise for the lower bound/minimum… but reversed.

Find the area under the curve $f(x) = x^2$ on the interval $[1,4]$ with $n = 5$ rectangles.

1. Find Δx. $\Delta x = \frac{b-a}{n}$. Here: $\Delta x = \frac{4-1}{5} = \frac{3}{5} = 0.6$
2. Create the partition: list the x_i values. $\{1, 1.6, 2.2, 2.8, 3.4, 4\}$
3. Select the values you will use and find the height of the function at those points (height of the rectangles). For the right-hand rule: $\{1.6, 2.2, 2.8, 3.4, 4\}$ Heights: $\{f(1.6), f(2.2), f(2.8), f(3.4), f(4)\} = \{2.56, 4.84, 7.84, 11.56, 16\}$
4. Find the area of each rectangle: $f(x_i)\Delta x$: $\{2.56(0.6), 4.84(0.6), 7.84(0.6), 11.56(0.6), 16(0.6)\} = \{1.536, 2.904, 4.704, 6.936, 9.6\}$
5. Add up. $1.536 + 2.904 + 4.704 + 6.936 + 9.6 = 25.68$

This is my estimate. (this is an over-estimate).

(True value is 21.)

For the left-hand rule:

1. Find Δx. $\Delta x = \frac{b-a}{n}$. Here: $\Delta x = \frac{4-1}{5} = \frac{3}{5} = 0.6$
2. Create the partition: list the x_i values. $\{1, 1.6, 2.2, 2.8, 3.4, 4\}$
3. Select the values you will use and find the height of the function at those points (height of the rectangles). For the right-hand rule: $\{1,1.6, 2.2, 2.8, 3.4\}$ Heights: $\{f(1), f(1.6), f(2.2), f(2.8), f(3.4)\} = \{1,2.56, 4.84,7.84,11.56\}$
4. Find the area of each rectangle: $f(x_i)\Delta x$: $\{1(0.6), 2.56(0.6), 4.84(0.6), 7.84(0.6), 11.56(0.6)\} = \{0.6, 1.536, 2.904,4.704,6.936\}$
5. Add up. $0.6 + 1.536 + 2.904 + 4.704 + 6.936 = 16.68$

As n gets bigger, the sum converges (gets closer and closer to) the true area. If we can come up with a formula in terms of n, and then let the limit go to infinity.

Steps for generic n.

1. Find Δx. $\Delta x = \frac{4-1}{n} = \frac{3}{n}$
2. Create the partition. $x_i = a + i\Delta x = 1 + i\left(\frac{3}{n}\right) = 1 + \frac{3i}{n}$
3. The sum starts at $i = 1$ for the right-hand rule, and ends at n. The sum starts at i=0, and ends at n-1 for the left-hand rule.

$$\sum_{i=1}^{n} f(x_i)\Delta x$$

Or

$$\sum_{i=0}^{n-1} f(x_i)\Delta x = \sum_{j=1}^{n} f(x_{j-1})\Delta x$$

4. Substitute into function.

$$\sum_{i=1}^{n} f\left(1 + \frac{3i}{n}\right)\left(\frac{3}{\text{n}}\right) = \sum_{i=1}^{n} \left(1 + \frac{3i}{n}\right)^2 \left(\frac{3}{n}\right)$$

Do algebra.

$$\left(1 + \frac{3i}{n}\right)^2 = 1 + \frac{6i}{n} + \frac{9i^2}{n^2}$$

$$\left(1 + \frac{3i}{n}\right)^2 \left(\frac{3}{n}\right) = \left(1 + \frac{6i}{n} + \frac{9i^2}{n^2}\right)\left(\frac{3}{n}\right) = \frac{3}{n} + \frac{18i}{n^2} + \frac{27i^2}{n^3}$$

$$\sum_{i=1}^{n} \left(1 + \frac{3i}{n}\right)^2 \left(\frac{3}{n}\right) = \sum_{i=1}^{n} \frac{3}{n} + \frac{18i}{n^2} + \frac{27i^2}{n^3} = \frac{3}{n}\sum_{i=1}^{n} 1 + \frac{18}{n^2}\sum_{i=1}^{n} i + \frac{27}{n^3}\sum_{i=1}^{n} i^2$$

$$\frac{3}{n}(n)+\frac{18}{n^2}\frac{\left(n(n+1)\right)}{2}+\frac{27}{n^3}\frac{n(n+1)(2n+1)}{6}=$$

$$3+\frac{9(n^2+n)}{n^2}+\frac{9}{2n^3}(2n^3+3n^2+n)=3+\frac{9n^2}{n^2}+\frac{9n}{n^2}+\frac{18n^3}{2n^3}+\frac{27n^2}{2n^3}+\frac{9n}{2n^3}$$

$$3+9+\frac{9}{n}+9+\frac{27}{2n}+\frac{9}{2n^2}$$

At this point, if you had a large n, you could substitute to match the estimate from the spreadsheet.

5. To get the exact value, take the limit as n goes to infinity.

$$\lim_{n\to\infty} 21+\frac{9}{n}+\frac{27}{2n}+\frac{9}{2n^2}=21$$

So, the true area of the region is

$$A=\lim_{n\to\infty}\sum_{i=1}^{n} f(x_i)\Delta x$$

The punchline:

We can calculate areas more easily:

Area under the curve is given by an integral:

$$A=\int_a^b f(x)dx=F(b)-F(a)$$

This called the Fundamental Theorem of Calculus. (the definite integral)

$$\int_1^4 x^2dx=\left.\frac{x^3}{3}\right|_1^4=\frac{(4)^3}{3}-\frac{(1)^3}{3}=\frac{64}{3}-\frac{1}{3}=\frac{63}{3}=21$$

Continue with 5.2 on Thursday.