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6.6 Centers of Mass
6.7 Integrals with Exponentials and Logs
6.8 Growth and Decay
6.9 Hyperbolic Trig Functions

Centers of Mass

Discrete case:
Moment of mass is the product of the mass times the distance (in whatever frame you are measuring in). If you have multiple masses you add them up. To locate the center of mass, you divide the result by the total mass. Results in a distance value that is center of mass.

Continuous case:
The total mass is going to depend on the shape of the region, and the density function.
In a previous chapter we had a density function $\rho(x)$, and a single dimensional distance, and then the total mass was $M = \int_a^b \rho(x)\, dx$.
In this section we are going to assume constant density, and the shape of the region will change to 2-d (area between curves). In this case ρ is a constant. The shape of region changes the amount of mass at any point away from the origin.

$$M = \rho \int_a^b [f(x) - g(x)]dx$$

Moments of mass also is obtained from integration.
Moment of mass from the y-axis, a distance in the x-direction.

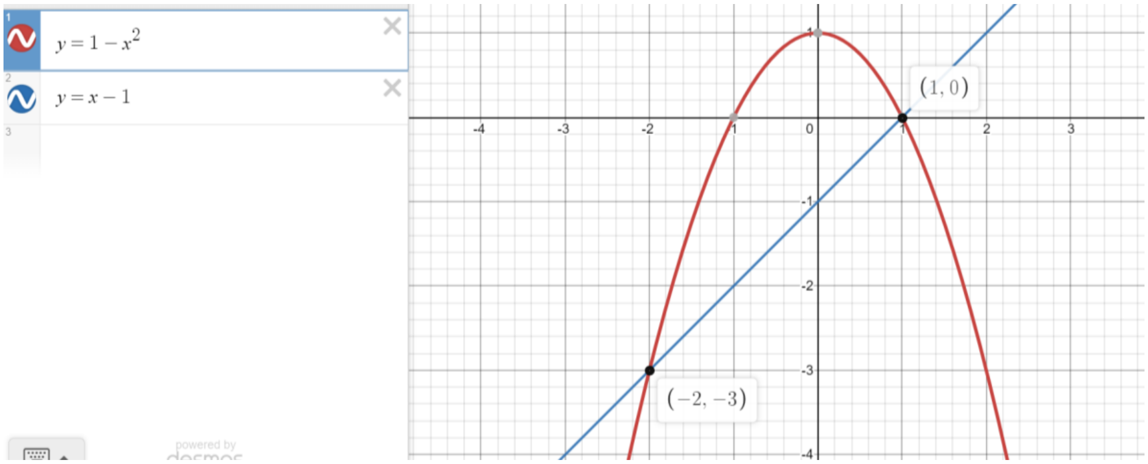
$$M_y = \rho \int_a^b x\,[f(x) - g(x)]dx$$

Moment of mass from the x-axis is a distance in y-direction.

$$M_x = \rho \int_a^b \frac{1}{2}[f^2(x) - g^2(x)]dx$$

Center (centroid) is $(\bar{x}, \bar{y}) = \left(\frac{M_y}{M}, \frac{M_x}{M}\right)$

Find the center of mass of the lamina bounded by $y = 1 - x^2$, and $y = x - 1$ assuming constant density.



$$M = \rho\int_{-2}^{1}(1-x^2)-(x-1)dx = \rho\int_{-2}^{1} 2-x^2-xdx = \rho[2x-\frac{1}{3}x^3-\frac{1}{2}x^2\Big|_{-2}^{1} =$$

$$\rho\left[2-\frac{1}{3}-\frac{1}{2}-\left(-4-\frac{1}{3}(-8)-\frac{1}{2}(4)\right)\right] = \rho\left[2+4-\frac{1}{3}-\frac{8}{3}-\frac{1}{2}+\frac{4}{2}\right] = \rho\left[6-3+\frac{3}{2}\right] = \rho\left[3+\frac{3}{2}\right] = \frac{9\rho}{2}$$

$$M_y = \rho\int_{-2}^{1} x\,[2-x^2-x]dx = \rho\int_{-2}^{1} 2x-x^3-x^2dx = \rho[x^2-\frac{1}{4}x^4-\frac{1}{3}x^3\Big|_{-2}^{1} =$$

$$\rho\left[1-\frac{1}{4}-\frac{1}{3}-\left(4-\frac{1}{4}(16)-\frac{1}{3}(-8)\right)\right] = \rho\left[-3-\frac{1}{4}+\frac{16}{4}-\frac{1}{3}-\frac{8}{3}\right] = \rho\left[-3+\frac{15}{4}-3\right] = \rho\left(-6+\frac{15}{4}\right) = -\frac{9\rho}{4}$$

$$M_x = \rho\int_{-2}^{1}\frac{1}{2}[(1-x^2)^2-(x-1)^2]dx = \frac{\rho}{2}\int_{-2}^{1} 1-2x^2+x^4-(x^2-2x+1)dx =$$

$$\frac{\rho}{2}\int_{-2}^{1} x^4-3x^2+2xdx = \frac{\rho}{2}[\frac{1}{5}x^5-x^3+x^2\Big|_{-2}^{1} = \frac{\rho}{2}\left[\frac{1}{5}-1+1-\left(\frac{1}{5}(-32)-(-8)+4\right)\right] =$$

$$\frac{\rho}{2}\left[\frac{1}{5}+\frac{32}{5}-12\right] = \frac{\rho}{2}\left[\frac{33}{5}-12\right] = \frac{\rho}{2}\left[-\frac{27}{5}\right] = -\frac{27\rho}{10}$$

Center of Mass:

$$(\bar{x},\bar{y}) = \left(\frac{M_y}{M},\frac{M_x}{M}\right) = \left(\frac{-\frac{9\rho}{4}}{\frac{9\rho}{2}},\frac{-\frac{27\rho}{10}}{\frac{9\rho}{2}}\right) = \left(-\frac{9}{4}\times\frac{2}{9},-\frac{27}{10}\times\frac{2}{9}\right) = \left(-\frac{1}{2},-\frac{3}{5}\right)$$

This location makes sense if we plot it on the graph.

Integrals of exponentials and logs

1. Remember the rules for integration
2. It can worth simplifying expressions using log or exponential rules before integrating.
3. Think about the substitution examples we did in Chapter 5 in terms of figuring out what to make u.

Exponential Growth and Decay
Reminder:

$$P(t) = P_0 e^{kt}$$

P_0 is the initial value
k is the rate of growth/decay. It is positive if the population is increasing, and negative if it's decreasing.

Rate of growth or decay (instantaneous) at a particular moment in time, take the derivative.

If my function is the rate of change, then integral of that rate will give me the accumulated growth (or decay).

Math of finance questions, compounded.

$$A(t) = P\left(1 + \frac{r}{n}\right)^{nt}$$

n is the number of times we are compounding each year.
r is annual interest rate
P is the principal
t is time in years

Continuous compounding.

$$A(t) = Pe^{rt}$$

Limit of the discrete compounding case, where r is an uncompounded annual interest (10%, then r=0.1).

They treat r as k, and we have to find it the same way we'd find any other k.

So if they say 10% annual interest, then what they mean is $A(1) = P \times 1.10$

$$(1.10)P = Pe^{r}$$
$$1.10 = e^{r}$$
$$\ln(1.10) = r$$

0.0953…

Example.
A 25-year-old student invests money into an account that pays 5% compounded continuously. How much does the student need to invest to get to 1 million dollars by the time they are 65?

A(40)=1,000,000
P?
R=0.05 or R=ln(1.05)

$$1{,}000{,}000 = Pe^{0.05(40)}$$

$$\frac{1{,}000{,}000}{e^{0.05(40)}} = 13{,}533.53$$

In some calculus books, they talk about as a way of introducing students to differential equations.

Exponential growth results from a linear multiplier of the function.

$$\frac{dP}{dt} = kP$$

$$P(t) = e^{kt}$$

$$P'(t) = \frac{dP}{dt} = ke^{kt}$$

Separation of variables:
We put all the function variables on one side, and all the dependent variables on the other.

$$\frac{dP}{P} = kdt$$

The trick is to integrate both sides.

$$\int \frac{dP}{P} = \int kdt$$

$$\ln P = kt + C$$

$$e^{\ln P} = e^{kt+C} = e^{kt} \times e^{C}$$

$$P(t) = P_0 e^{kt}$$

End of 6.9—hyperbolic trig functions.