4/5/2022

5.5 Substitution
5.6 Integrals involving logs and exponentials
5.7 Integrals involving inverse trig functions

Exam 2 is when? April 14th – next Thursday
Tuesday 12, we will spend on Review

Substitution is a method for dealing with the derivative of composite functions: reversing the chain rule.

There is u-sub handout on canvas.

$$\int f(g(x))g'(x)dx$$

$$\int (x^3 - 3x + 1)^4(3x^2 - 3)dx$$

$$\int xe^{x^2}dx$$

Looking at two functions (products) inside an integral, where one function is the derivative of some part of the other function (composition), up to a constant multiplier.

Examples.

$$\int 6x(3x^2 + 4)^5dx$$

We want to identify the composed function, and the part that is the result of the chain rule. Composed function as f(u), and du as the chain rule portion.

$$f(u) = (3x^2 + 4)^5 = u^5$$
$$u = 3x^2 + 4$$
$$du = 6xdx$$

If I take the derivative of u do I get 6x? In this case, the answer is yes. That confirms my choice of u as being correct. If there is constant multiplier missing, we can fix it here.

Substitute into the integral: replace all x's with expressions in u.

$$\int (3x^2 + 4)^5(6xdx) = \int u^5du = \frac{1}{6}u^6 + C = \frac{1}{6}(3x^2 + 4)^6 + C$$

Example.

$$\int z\sqrt{z^2 - 5}dz$$

$$f(u) = \sqrt{z^2 - 5} = u^{1/2}$$

$$u = z^2 - 5$$

$$du = 2zdz$$

We want to match what is in the problem, which is zdz, not 2zdz. So divide this equation by 2.

$$\frac{1}{2}du = zdz$$

$$\int z\sqrt{z^2 - 5}dz = \int \sqrt{z^2 - 5}(zdz) = \int u^{\frac{1}{2}}\left(\frac{1}{2}du\right) = \frac{1}{2}\int u^{\frac{1}{2}}du = \frac{1}{2}\frac{2}{3}u^{3/2} + C = \frac{1}{3}(z^2 - 5)^{3/2} + C$$

Put the original variable back when you are done.

$$\int \tan x \, dx = \int \frac{\sin x}{\cos x} dx$$

$$\int \frac{g'(x)}{g(x)} dx$$

$$\frac{d}{dx}\left[\ln\big(g(x)\big)\right] = \frac{1}{g(x)} \times g'(x)$$

u substitution when you have a ratio without a power, the denominator is u.

$$u = \cos x$$
$$du = -\sin x \, dx$$
$$-du = \sin x \; dx$$

$$\int \frac{\sin x}{\cos x} dx = \int -\frac{du}{u} = -\int \frac{1}{u} du = -\ln|u| + C = -\ln|\cos x| + C = \ln|\sec x| + C$$

Example.

$$\int \frac{\sin x}{\cos^3 x} dx = \int \sin x \; (\cos x)^{-3} dx$$

Similar substitution, but integrate with power rule rather than log rule.

$$u = \cos x$$
$$du = -\sin x \, dx$$
$$-du = \sin x \, dx$$

$$\int \frac{\sin x}{\cos^3 x} dx = \int \sin x \; (\cos x)^{-3} dx = \int u^{-3}(-du) = -\int u^{-3} du = -\left(-\frac{1}{2}u^{-2}\right) + C$$
$$= \frac{1}{2}(\cos x)^{-2} + C = \frac{1}{2}\sec^2 x + C$$

Example.

$$\int xe^{4x^2+3}dx$$

For an exponential function, think about the exponent as being your u.

$$u = 4x^2 + 3$$
$$du = 8xdx$$
$$\frac{1}{8}du = xdx$$

$$\int xe^{4x^2+3}dx = \int e^u\left(\frac{1}{8}du\right) = \frac{1}{8}\int e^u du = \frac{1}{8}e^u + C = \frac{1}{8}e^{4x^2+3} + C$$

$$\frac{d}{dx}\left[\frac{1}{8}e^{4x^2+3}\right] = \frac{1}{8}e^{4x^2+3}(8x) = xe^{4x^2+3}$$

Example.

$$\int e^x\sqrt{1-e^x}dx$$

$$u = 1 - e^x$$
$$du = -e^x dx$$

$$-du = e^x dx$$

$$\int e^x\sqrt{1-e^x}dx = \int u^{1/2}(-du) = -\int u^{\frac{1}{2}}du = -\frac{2}{3}u^{\frac{3}{2}} + C = -\frac{2}{3}(1-e^x)^{\frac{3}{2}} + C$$

Example.

$$\int e^{2x}dx$$

$$u = 2x$$
$$du = 2dx$$
$$\frac{1}{2}du = dx$$

$$\int e^{2x}dx = \int e^u\left(\frac{1}{2}du\right) = \frac{1}{2}\int e^u du = \frac{1}{2}e^u + C = \frac{1}{2}e^{2x} + C$$

Example.

$$\int \frac{e^{\sqrt{x}}}{\sqrt{x}}dx = \int e^{x^{1/2}}x^{-\frac{1}{2}}dx$$

$$u = x^{\frac{1}{2}}$$
$$du = \frac{1}{2}x^{-\frac{1}{2}}dx$$

$$2du = x^{-\frac{1}{2}}dx$$

$$\int \frac{e^{\sqrt{x}}}{\sqrt{x}}dx = \int e^{x^{1/2}}x^{-\frac{1}{2}}dx = \int e^{u}(2du) = 2\int e^{u}du = 2e^{u} + C = 2e^{\sqrt{x}} + C$$

Example.
(Double Substitution)

$$\int \frac{1}{x\ln x\ \ln(\ln x)}dx$$

$$\mathrm{u} = \ln(\ln x)$$
$$du = \frac{1}{\ln x} \times \frac{1}{x}dx = \frac{1}{x\ln x}dx$$

$$\int \frac{1}{\ln(\ln x)}\left(\frac{1}{x\ln x}dx\right) = \int \frac{1}{u}du = \ln|u| + C = \ln(\ln(\ln x)) + C$$

This is the fast way. Pick the function that is the messiest. This avoids the double substitution.

Suppose you pick $u = \ln x$ as your substitution?

$$u = \ln x$$
$$du = \frac{1}{x}dx$$

$$\int \frac{1}{x\ln x\ \ln(\ln x)}dx = \int \frac{1}{u\ln(u)}du$$

Second substitution

$$v = \ln u$$
$$dv = \frac{1}{u}du$$

$$\int \frac{1}{u\ln(u)}du = \int \frac{1}{v}dv = \ln v + C = \ln(\ln u) + C = \ln(\ln(\ln x)) + C$$

Another example:

$$\int \frac{e^{2\ln(1-t)}}{1-t}dt$$

Alternative substitutions. Change of variable.

$$\int \cos^2 x\, dx$$

Use the power-reducing identifies for sine or cosine squared.

$$\cos^2 x = \frac{1}{2}(1 + \cos 2x)$$

$$\sin^2 x = \frac{1}{2}(1 - \cos 2x)$$

$$\int \cos^2 x\, dx = \int \frac{1}{2}(1 + \cos 2x)dx = \int \frac{1}{2}dx + \frac{1}{2}\int \cos 2x\, dx$$

Finish by regular substitution (see example with e^{2x})

Change of variables.
Tends to work with functions that products, one is a root, and the other is not the derivative of the inside.

$$\int x\sqrt{1-x}dx$$

The difference here is the functions are not the result of a chain rule. $u = \sqrt{1-x}$
The other difference is that you are going to solve for x before taking the derivative.

$$u^2 = 1 - x$$
$$x = 1 - u^2$$
$$dx = -2udu$$

$$\int x\sqrt{1-x}dx = \int (1-u^2)u(-2udu) = -2\int u^2 - u^4 du = -2\left[\frac{u^3}{3} - \frac{u^5}{5}\right] + C$$

$$-\frac{2}{3}(1-x)^{\frac{3}{2}} + \frac{2}{5}(1-x)^{\frac{5}{2}} + C$$

Next time: inverse trig functions and hyperbolic trig functions (6.9)