3/28/2023

Probability Rules and Notation
Counting
Probability from Tables
Conditional Probability

Don't forget about Project #4, due April 2nd

Probability
All probabilities are essentially proportions: they are ratios part/whole. All probabilities are between 0 and 1 (inclusive).
A probability that is equal to 0 means that the event is impossible.
A probability that is equal to 1 means that the event is certain (it must occur).

If I role a standard die, the probability of getting 7 is 0.
If I flip a coin that has heads on both sides, the probability of getting a head is 1.

The probabilities associated with any one random outcome must add to 1.
The probability of heads + probability of tails = 1 (100%)

Sample space: a set of all the possible outcomes of an event (random event)
The sample space for a coin flip {H,T}
The sample for a standard die role is {1, 2, 3, 4, 5, 6}

The sample space for two coin flips {HH, HT, TH, TT}

The sample space is composed of a list of all the simple events. Not compound events.

Events in the sample space are assumed to be "equiprobable" or equally likely outcomes. Each thing in the sample space has the same probability. The number of events in the sample space is the denominator in our probability calculations.

In the world of classical or theoretical probabilities, we are assuming that every event in the sample space is equally likely, and this will reduce our probability calculations to essentially just counting. The number of ways the "event" that we are interested in can occur divided by the number of total things that can occur in the sample space.

When we talk about probability, we use the notation $P(x)$ where x is the event in question, such as P(1) may be interpreted as the probability that the outcome equals 1 (let's say in a die role). P(H) is the probability of getting a head. Or you may see P(heads), or $P(X = 1)$.

The sample space for two coin flips is {HH, HT, TH, TT}
What is the probability of getting 2 heads: P(2 heads) or P(HH)?

$$P(2\ heads) = \frac{1}{4}$$

What is the probability of getting two tails?

$$P(2\ tails) = P(0\ heads) = \frac{1}{4}$$

What is the probability of getting 1 head?

$$P(1\ head) = \frac{2}{4} = \frac{1}{2}$$

$$\frac{1}{4} + \frac{1}{4} + \frac{2}{4} = \frac{4}{4} = 1$$

In a sample space, all the events in the list must be mutually exclusive.

History/Regions	East	Midwest	South	West	Grand Total
1	58	57	63	52	230
2	52	63	47	50	212
3	63	59	61	72	255
NA	79	82	82	60	303
Grand Total	**252**	**261**	**253**	**234**	**1000**

What is the probability **that someone randomly selected from this dataset** has no previous purchase history with the company?

$$\frac{303}{1000}$$

What is the probability the someone lives in the Midwest?

$$\frac{261}{1000}$$

If I want to find the probability that someone live BOTH in the Midwest AND has no previous purchase history, this is an intersection question.

$$P(MidWest\ and\ NA) = P(MidWest\ \cap NA) = \frac{82}{1000}$$

The probability of being EITHER in the Midwest OR having no previous purchase history

$$P(Midwest\ OR\ NA) = P(Midwest\ \cup NA) = P(Midwest) + P(NA) - P(Midwest\ AND\ NA) =$$

$$\frac{261}{1000} + \frac{303}{1000} - \frac{82}{1000} = \frac{261 + 303 - 82}{1000} = \frac{482}{1000}$$

General formula:

$$P(A\ or\ B) = P(A) + P(B) - P(A\ and\ B)$$

Conditional probability:
$P(A|B) =$ probability of A given event B has already happened

If two events are independent of each other, then knowing that B happened does not change the probability of A.

$$P(A) = P(A|B)$$

In most real world data, this is not true. In most cases, the data, even if it's only in a very small way, are dependent, and so this property is not the case.

What is the probability (from this data) that someone lives in the Midwest **given** that they have no previous purchase history?

$$\frac{82}{303}$$

P(Midwest)=261/1000=26.1%
P(Midwest|NA)=82/303=27.06…%

These probabilities are NOT the same, so these variables (region and purchase history) are dependent.

What is the probability that someone has no previous purchase history **given** that they are from the Midwest?

$$\frac{82}{261}$$

P(NA) = 303/1000 = 30.3%
P(NA|Midwest) = 82/261 = 31.417…%

Counting Rules

Multiplication Rule
Permutations
Combinations

Multiplication Rule: occurs each component of an event is counted independently of other elements of the event.

Consider a state with license plates of the form AAA-999 (three letters, followed by three numbers)
There are 26 letters in the English alphabet. And there are 10 single digit numbers. How many possible license plates are there?

$$26 \times 26 \times 26 \times 10 \times 10 \times 10 = 16{,}900{,}000$$

If a state allowed license plates with 6 digits, but any digit could be either a letter or a number: there would be 36 options for each position.

$$36^6 = 2{,}176{,}782{,}336$$

If I flip a coin 10 times, how many possible outcomes are there?

$$2^{10} = 1024$$

Permutations
Do not allow us to repeat any value over again.

Suppose there are 14 players on a t-ball team, and you want to put 10 on the field in various positions: first, pitcher, etc.

14x13x12x11x10x9x8x7x6x5=3,632,428,800

Any scenario in which the order matters, but values can't repeat, is a permutation.

A slate of officers is ranked (order matters) and no repetition (can't hold two offices).

Factorial: 6!
6x5x4x3x2x1=720
7! = 7x6x5x4x3x2x1=5040

Permutation notation examples:

$$P(n, r) = nPr = {}^{n}_{r}P$$

n is the number of elements we are choosing from, and r is the number of elements being chosen.

Our baseball problem: $P(14,10) = 14P10$

$$nPr = \frac{n!}{(n-r)!}$$

$$14P10 = \frac{14!}{(14-10)!} = \frac{14 \times 13 \times 12 \times 11 \times 10 \times 9 \times 8 \times 7 \times 6 \times 5 \times 4 \times 3 \times 2 \times 1}{4 \times 3 \times 2 \times 1} =$$

$$14 \times 13 \times 12 \times 11 \times 10 \times 9 \times 8 \times 7 \times 6 \times 5 = 3{,}632{,}428{,}800$$

Combinations
Order doesn't matter; no repetition

Couples
Prizes (raffle) where all the values are the same … everyone gets the same bonus
Serving on a committee or jury
Number of heads in a given number of flips

Suppose I flip a coin 15 times, and I want to know the number of ways I can get 7 heads.
Suppose I have 10 colleagues in my department and 5 of us are randomly selected to serve on a committee. How many different committees could be formed?

$$C(n,r) = nCr = {}_r^nC = \binom{n}{r}$$

$$C(n,r) = nCr = {}_r^nC = \binom{n}{r} = \frac{n!}{(n-r)!\,r!}$$

Coin flip example: n is the number of flips (15), and r is the number of heads (7)

$$15C7 = \binom{15}{7} = \frac{15!}{(15-7)!\,7!} = \frac{15 \times 14 \times 13 \times 12 \times 11 \times 10 \times 9 \times 8 \times 7 \times 6 \times 5 \times 4 \times 3 \times 2 \times 1}{8 \times 7 \times 6 \times 5 \times 4 \times 3 \times 2 \times 1 \times 7 \times 6 \times 5 \times 4 \times 3 \times 2 \times 1} =$$

$$\frac{15 \times 14 \times 13 \times 12 \times 11 \times 10 \times 9}{7 \times 6 \times 5 \times 4 \times 3 \times 2 \times 1} = \frac{2 \times 13 \times 2 \times 11 \times 5 \times 9}{4 \times 1} = \frac{13 \times 11 \times 5 \times 9}{1} = 6435$$

Card games: a hand is a set of cards but the value of the cards does not depend on the order.

See Excel for how to calculate in Excel.

How do we use these in probability calculations?

Please read questions about counting and probability very carefully.

How many different ways can… is asking for a count = large number
What is the probability of… is asking for a proportion = small number

Suppose you flip a coin 12 times and want to know the probability of getting 8 heads.
You need 2 counts to calculate the probability. You need to the total number of ways to flip a coin 12 times for the denominator. You need the number of ways of getting 8 heads in 12 flips for the numerator.

$$\frac{12C8}{2^{12}} = \frac{495}{4096}$$

Suppose a state has a license plate scheme with AAA-999 as the general pattern. What is the probability that you will get only vowels and even numbers in your plate?

$$\frac{5 \times 5 \times 5 \times 5 \times 5 \times 5}{26^3 \times 10^3} = 8.89 \times 10^{-4} = 0.000889 \ldots$$

In a 5-card poker hand, what is the probability of getting a full house with 2 kings and 3 aces?

$$\frac{\binom{4}{2}\binom{4}{3}}{\binom{52}{5}} = \frac{24}{2598960} = 9.23 \times 10^{-6}$$

What if we wanted to figure in other full houses?

$$13(12)\frac{\binom{4}{2}\binom{4}{3}}{\binom{52}{5}} = 0.00144\,\ldots$$