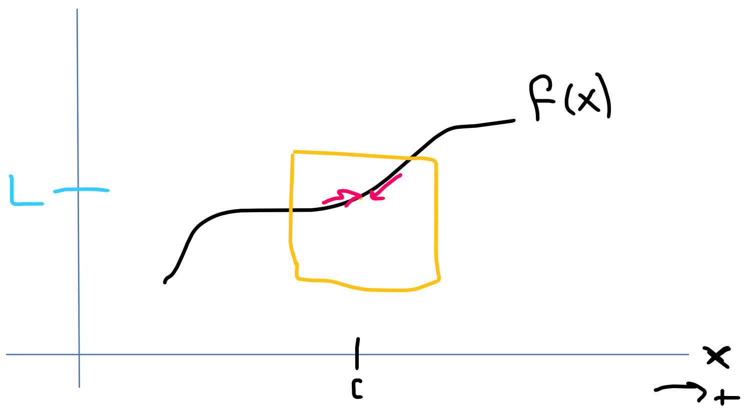
1/24/2024

Limits: Numerical, Algebraic, Graphical
Continuity
Average Rate of Change
Definition of the Derivative

Limits
The value of the function approaches "the limit" as the x-value approaches a given number (c).

$$\lim_{x \to c} f(x) = L$$

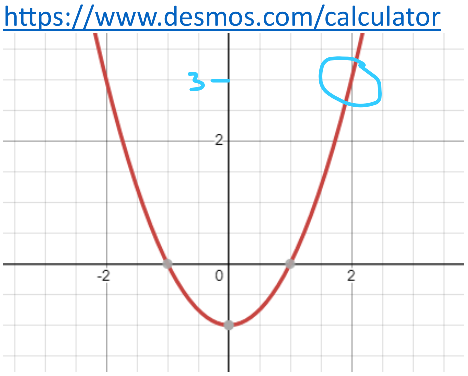


If the function is continuous and well-defined, then the value L may be equal to $f(c)$, the value of the function at that point.

Simple example.
What is the value of $\lim_{x \to 2} x^2 - 1 =?$

$$\lim_{x \to 2} x^2 - 1 = (2)^2 - 1 = 3$$

https://www.desmos.com/calculator



x	x^2-1
3	8
2.1	3.41
2.01	3.0401

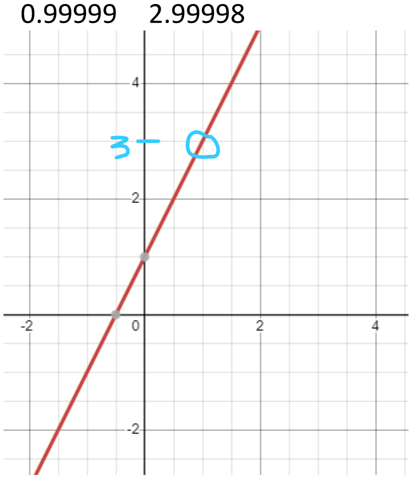
2.001	3.004001
2.0001	3.0004
2.00001	3.00004
1	0
1.9	2.61
1.99	2.9601
1.999	2.996001
1.9999	2.9996
2	3

Find $\lim_{x \to 1} \frac{2x^2-x-1}{x-1}$

$$\frac{2(1)^2-1-1}{1-1} = \frac{0}{0}$$

Division by 0 is not defined.

x	f(x)
1	#DIV/0!
2	5
1.1	3.2
1.01	3.02
1.001	3.002
1.0001	3.0002
1.00001	3.00002
0	1
0.9	2.8
0.99	2.98
0.999	2.998
0.9999	2.9998
0.99999	2.99998



$x = 1$ produces a hole in the graph

Algebraic: simplify the expression if possible and then plugging the value back into the simplified expression.

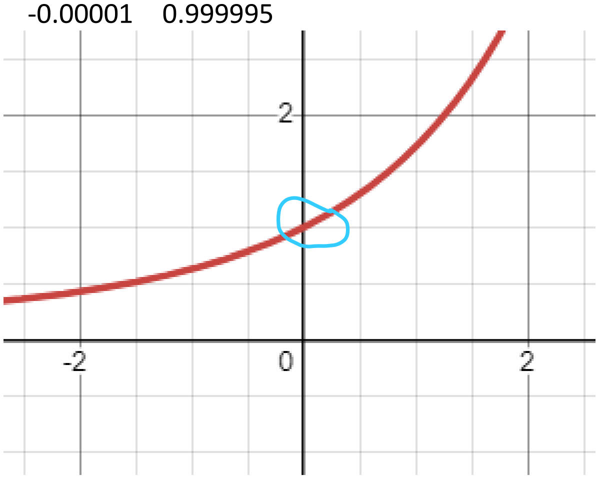
$$\lim_{x \to 1} \frac{2x^2 - x - 1}{x - 1} = \lim_{x \to 1} \frac{(2x+1)(x-1)}{x-1} = \lim_{x \to 1} 2x + 1 = 2(1) + 1 = 3$$

For rational functions in particular, if the limit produces 0/0, then factor and cancel common factors. Then try plugging into the reduced function.

Find $\lim_{x \to 0} \frac{e^x - 1}{x}$

$$\lim_{x \to 0} \frac{e^x - 1}{x} = \frac{e^0 - 1}{0} = \frac{1-1}{0} = \frac{0}{0}$$

x	f(x)
0	#DIV/0!
1	1.718282
0.1	1.051709
0.01	1.005017
0.001	1.0005
0.0001	1.00005
0.00001	1.000005
-1	0.632121
-0.1	0.951626
-0.01	0.995017
-0.001	0.9995
-0.0001	0.99995
-0.00001	0.999995



One-sided limits

Approaching the value from the left side (from values smaller than the target x value)

$$\lim_{x \to c^-} f(x)$$

Approaching the value from the right side (from values larger than the target x value)

$$\lim_{x \to c^+} f(x)$$

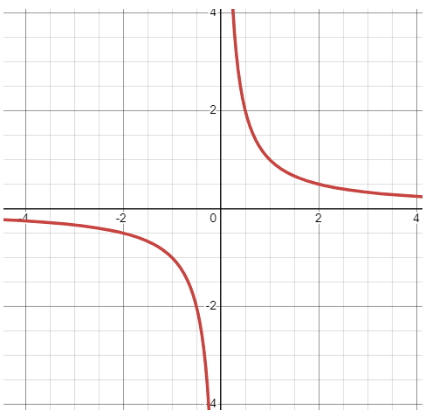
Consider the function $f(x) = \sqrt{x}$

$$\lim_{x \to 0^+} \sqrt{x} = 0$$

Consider the function $f(x) = \frac{1}{x}$

$$\lim_{x \to 0^-} \frac{1}{x} = -\infty$$

$$\lim_{x \to 0^+} \frac{1}{x} = \infty$$



x	1/x
0	#DIV/0!
0.1	10
0.01	100
0.001	1000
0.00001	100000
-0.1	-10
-0.01	-100
-0.001	-1000
-0.0001	-10000

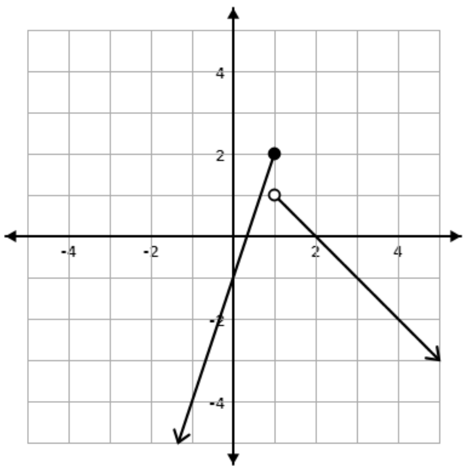
Since the two sides are not going to the same place,

$$\lim_{x \to 0} \frac{1}{x} = DNE$$

We say it "does not exist" because the two sides are unequal.

$$\lim_{x \to c^-} f(x) \neq \lim_{x \to c^+} f(x) \to \lim_{x \to c} f(x) = DNE$$

Piecewise functions are where the one-sided come up the most often



https://www.graphfree.com/grapher.html

$$f(x) = \begin{cases} 3x - 1, & x \leq 1 \\ 2 - x, & x > 1 \end{cases}$$

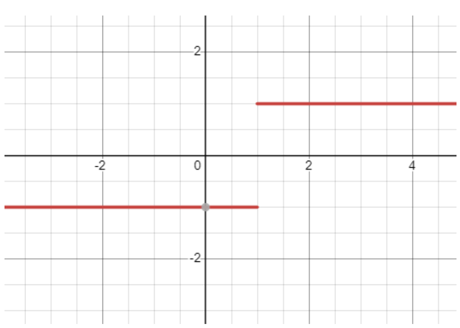
What is the limit (if it exists) at x=1?

$$\lim_{x \to 1^-} f(x) = \lim_{x \to 1^-} 3x - 1 = 3(1) - 1 = 2$$

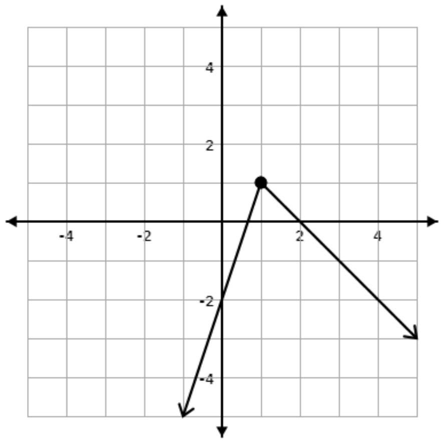
$$\lim_{x \to 1^+} f(x) = \lim_{x \to 1^+} 2 - x = 2 - 1 = 1$$

The limit at x=1 does not exist.

$$f(x) = \frac{|x - 1|}{x - 1}$$



the limit at x=1 does not exist



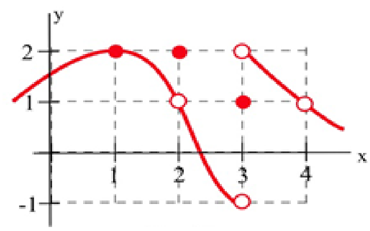
$$f(x) = \begin{cases} 3x - 2, & x \leq 1 \\ 2 - x, & x > 1 \end{cases}$$

$$\lim_{x \to 1} f(x) = 1$$

1. Use the graph to determine the following limits.

(a) $\lim_{x \to 1} f(x)$ (b) $\lim_{x \to 2} f(x)$

(c) $\lim_{x \to 3} f(x)$ (d) $\lim_{x \to 4} f(x)$



(from 2.2 of textbook)

Continuity

If you can draw the graph without picking up your pencil, the graph is continuous.

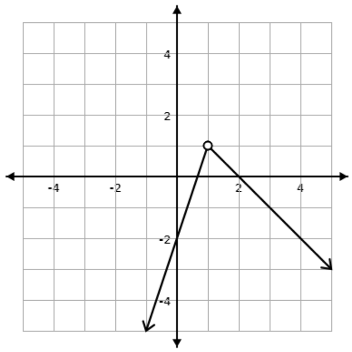
When a function is continuous, the limit exists at every point, and the function value and the limit agree.

There are no holes in the graph (no undefined points), there are no jumps in the graph, and no vertical asymptotes.

Three kinds of discontinuities:

- Point discontinuities (hole)
- Jump discontinuities (piecewise functions that don't join… a finite gap between the two pieces)
- Infinite discontinuity (the graph goes to infinity or negative infinity on either side of the point)

Point discontinuity: (removal discontinuity)

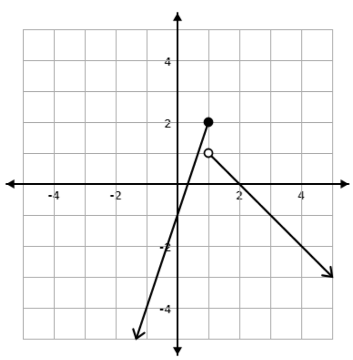


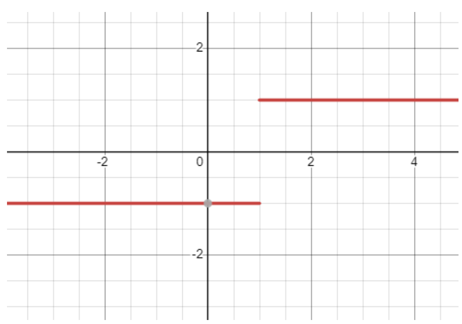
$$f(x) = \begin{cases} 3x - 2, & x < 1 \\ 2 - x, & x > 1 \end{cases}$$

Or $f(x) = \frac{2x^2 - x - 1}{x - 1}$

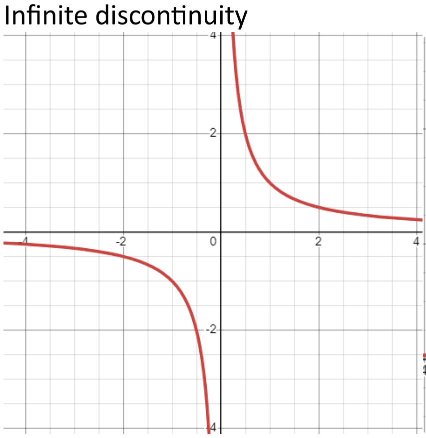
If you redefine the function at one specific point, you can fill in the hole. We can “repair” the function, so that it’s still a function, but is also continuous.

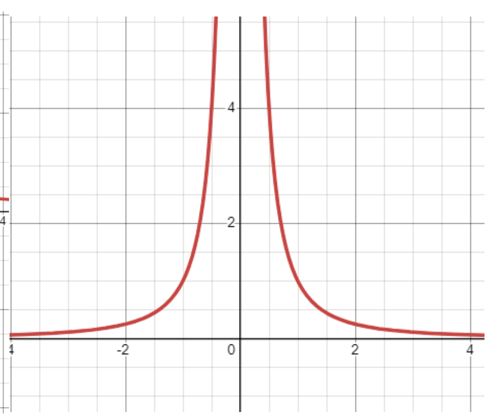
Jump discontinuity





Infinite discontinuity

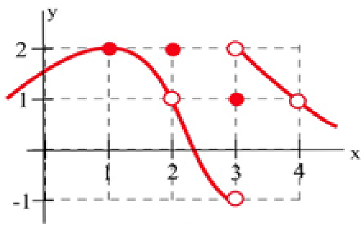




Any function that is missing all of these is a continuous function.

Continuity:

a) The left-hand limit and the right-hand limit exist
b) The limit exists (the left- and right-hand limit agree)
c) The function is defined at the point
d) The function and the limit have the same value



Determine the continuity at the points x=1, x=2, x=3, x=4.

X=3
We already said that the limit did exist here. The left-hand limit is -1, and the right-hand limit is 2. These are not equal, and so the limit doesn't exist. The function is not continuous at x=3. (jump discontinuity)

X=2
The limit does exist here. The function approaches 1 from both the left and the right, so the limit is equal to 1. $f(2) = 2$. Here, the limit and function value are not the same, and so the function is discontinuous at x=2. (removable, or point discontinuity)

X=4
The limit does exist, approaching 1 on both side. But the function is not defined there. There is a gap in the function and so it is discontinuous. (removable, or point discontinuity).

X=1
The limit exists and is equal to 2. The $f(1) = 2$ and since they agree, the function is continuous at that point.

Rates of change
=slopes

Recall the definition of the slope:

$$m = \frac{y_2 - y_1}{x_2 - x_1}$$

Find the slope of the line connecting the points $(2,4), (3,-7)$

$$m = \frac{-7-4}{3-2} = -\frac{11}{1} = -11$$

Suppose a particle is moving along a curve defined by $f(x) = 40\sqrt{x+9}$
Find the average rate of change between the points x=0, and x=16

$$f(0) = 40\sqrt{0+9} = 40(3) = 120$$
$$f(16) = 40\sqrt{16+9} = 40(5) = 200$$

Average rate of change is the slope of the straight line connecting those points:

$$m_{avg} = \frac{200 - 120}{16 - 0} = \frac{80}{16} = 5$$

Difference quotient

$$\frac{f(x+h) - f(x)}{h}$$

Find the difference quotient for $f(x) = x^2 + 1$

$$\frac{(x+h)^2 + 1 - (x^2+1)}{h} = \frac{(x+h)(x+h) + 1 - x^2 - 1}{h} = \frac{\cancel{x^2} + 2xh + h^2 \cancel{+1} - \cancel{x^2} \cancel{-1}}{h}$$

$$= \frac{2xh + h^2}{h} = \frac{h(2x+h)}{h} = 2x + h$$

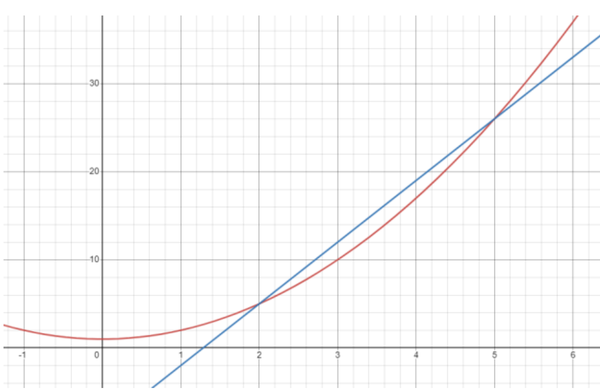
Suppose I want to find the average rate of change between x=2 and x=5
h is the difference between the two points, h=3

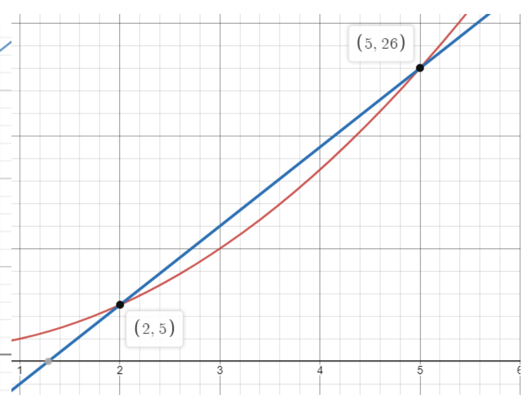
$$m_{avg} = 2(2) + 3 = 7$$

$$f(2) = 5, f(5) = 26$$

$$m_{avg} = \frac{26 - 5}{5 - 2} = \frac{21}{3} = 7$$

The line the connects two points on a curve is called the secant line. Average rate of change is also called the slope of the secant line.





The red line is $f(x) = x^2 + 1$
The blue line is the secant line $y = 7x - 9$

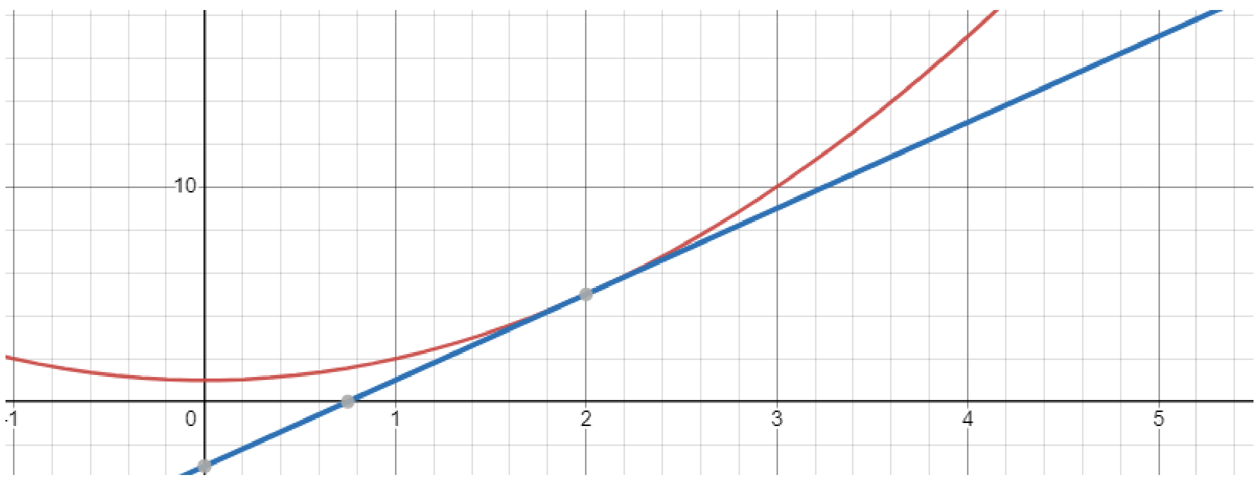
What happens as the two points that define the secant get closer together?

$$\lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

The expression produces the slope of the tangent line, touches the graph at only one point, but points in the direction of the instantaneous rate of change.

For our $f(x) = x^2 + 1$ example,

$$\lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \to 0} 2x + h = 2x$$



$$\lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

The definition of the derivative.

Find the derivative of the function $f(x) = 2x^2 - x - 1$ using the definition of the derivative.

$$\lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \to 0} \frac{2(x+h)^2 - (x+h) - 1 - (2x^2 - x - 1)}{h} =$$

$$\lim_{h \to 0} \frac{2(x^2 + 2xh + h^2) - (x+h) - 1 - (2x^2 - x - 1)}{h} =$$

$$\lim_{h \to 0} \frac{\cancel{2x^2} + 4xh + 2h^2 - \cancel{x} - h - \cancel{1} - \cancel{2x^2} + \cancel{x} + \cancel{1}}{h} =$$

$$\lim_{h \to 0} \frac{4xh + 2h^2 - h}{h} = \lim_{h \to 0} \frac{h(4x + 2h - 1)}{h} = \lim_{h \to 0} 4x + 2h - 1 = 4x - 1$$

Find the derivative of the function $f(x) = \frac{1}{x}$ using the limit definition of the derivative.

$$\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}=\lim_{h\to 0}\frac{\frac{1}{x+h}-\frac{1}{x}}{h}=\lim_{h\to 0}\left(\frac{1}{h}\right)\left(\frac{1}{x+h}-\frac{1}{x}\right)$$

For these kinds of problems, find a common denominator.

$$\lim_{h\to 0}\left(\frac{1}{h}\right)\left(\frac{1}{x+h}\left(\frac{x}{x}\right)-\frac{1}{x}\left(\frac{x+h}{x+h}\right)\right)=\lim_{h\to 0}\left(\frac{1}{h}\right)\left(\frac{x}{x(x+h)}-\frac{x+h}{x(x+h)}\right)=$$

$$\lim_{h\to 0}\left(\frac{1}{h}\right)\left(\frac{x-(x+h)}{x(x+h)}\right)=\lim_{h\to 0}\left(\frac{1}{h}\right)\left(\frac{\cancel{x}-\cancel{x}-h}{x(x+h)}\right)=\lim_{h\to 0}\left(\frac{1}{\cancel{h}}\right)\left(\frac{-\cancel{h}}{x(x+h)}\right)=$$

$$\lim_{h\to 0}\left(\frac{-1}{x(x+h)}\right)=-\frac{1}{x(x+0)}=-\frac{1}{x(x)}=-\frac{1}{x^2}=-x^{-2}$$

Find the derivative of $f(x)=\sqrt{x}$ using the limit definition.

$$\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}=\lim_{h\to 0}\frac{\sqrt{x+h}-\sqrt{x}}{h}=$$

The trick for the radical functions is to rationalize the numerator.

$$\lim_{h\to 0}\frac{\sqrt{x+h}-\sqrt{x}}{h}\times\frac{\sqrt{x+h}+\sqrt{x}}{\sqrt{x+h}+\sqrt{x}}=\lim_{h\to 0}\frac{\cancel{x}+h-\cancel{x}}{h\left(\sqrt{x+h}+\sqrt{x}\right)}=$$

$$\lim_{h\to 0}\frac{\cancel{h}}{\cancel{h}\left(\sqrt{x+h}+\sqrt{x}\right)}=\lim_{h\to 0}\frac{1}{\left(\sqrt{x+h}+\sqrt{x}\right)}=\frac{1}{\sqrt{x}+\sqrt{x}}=\frac{1}{2\sqrt{x}}$$