2/20/2024

Improper Integrals
Review for Exam #1

Improper Integrals:

1) If one of the limits is infinity (or both)
2) One of the limits (or both) is undefined at that value.
3) If there is a point of discontinuity inside the interval of integration (between the limits)

Some example of improper integrals:

$$\int_0^\infty xe^{-x}dx$$

$$\int_{-1}^1 \frac{1}{\sqrt{1-x^2}}dx$$

$$\int_{-1}^1 \frac{1}{\sqrt[3]{x^2}}dx$$

General idea is if the limit is a problem value, use a limit (from way back in Calc I) to replace the limit of the integral with a dummy value, take the antiderivative, and then evaluate the limit at the limiting value.
If both limits are a problem, then we split the integral into two pieces and deal with each endpoint separately.
If the problem point is inside the interval, we split the integral at the problem point, and then apply the same limit process to evaluate it.

In the case where we've split the integral, both pieces must be non-finite for there to be a value for the whole integral.

If any part of the integral goes to infinity, we say the integral is divergent, or it diverges. If both parts are finite, then we say the integral converges, and it converges to the value we found for the limit.

Example.

$$\int_0^\infty xe^{-x}dx$$

Replace infinity with b, and then take the limit as b goes to infinity.

$$\lim_{b\to\infty}\int_0^b xe^{-x}\,dx$$

Then do the integration.
For this one, we'll need integration by parts.

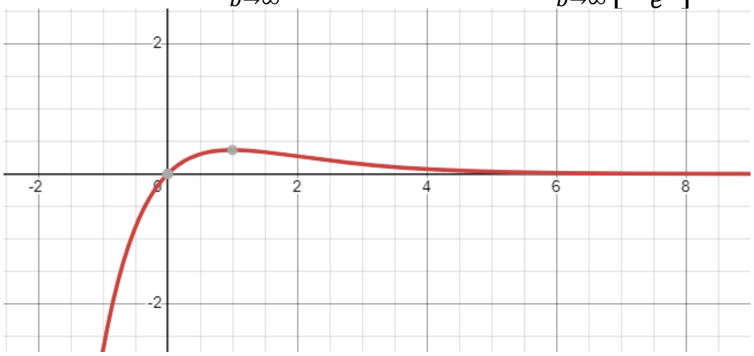
$$u = x, dv = e^{-x}dx$$

$$du = 1dx, v = -e^{-x}$$

$$\lim_{b\to\infty}\left[-xe^{-x} - \int_0^b -e^{-x}dx\right] = \lim_{b\to\infty}\left[-xe^{-x} + \int_0^b e^{-x}dx\right] =$$

$$\lim_{b\to\infty}[-xe^{-x} - e^{-x}]_0^b = \lim_{b\to\infty}[-be^{-b} - e^{-b}] + 0e^0 + e^0 =$$

$$1 + \lim_{b\to\infty}[-be^{-b} - e^{-b}] = 1 + \lim_{b\to\infty}\left[-\frac{b}{e^b}\right] = 1 + \lim_{b\to\infty}\left[-\frac{1}{e^b}\right] = 1 + 0 = 1$$



The area under the curve $f(x) = xe^{-x}$ is 1, on the interval $[0, \infty)$.
We say this integral converges.

Example.

$$\int_{-1}^{1}\frac{1}{\sqrt{1-x^2}}dx$$

Both endpoints of the interval are a problem because both produce division by 0 in the original function.

Split the integral in the middle of the interval (somewhere convenient) and then deal with one endpoint at a time.

$$\lim_{a\to-1}\int_a^0 \frac{1}{\sqrt{1-x^2}}dx + \lim_{b\to1}\int_0^b \frac{1}{\sqrt{1-x^2}}dx$$

$$\lim_{a\to-1}[\arcsin x]_a^0 + \lim_{b\to1}[\arcsin x]_0^b = \lim_{a\to-1}(\arcsin 0 - \arcsin a) + \lim_{b\to1}(\arcsin b - \arcsin 0) =$$

$$\lim_{a\to-1}(-\arcsin a) + \lim_{b\to1}(\arcsin b) = -\arcsin(-1) + \arcsin(1) = -\left(-\frac{\pi}{2}\right) + \frac{\pi}{2} = \pi$$

We also say that the integral is convergent.

Example.

$$\int_{-1}^{2}\frac{1}{\sqrt[3]{x^2}}dx$$

The problem with this integral is the 0 is inside the interval and 0 in the integrand makes it undefined.

Split the integral at the point where the function is not defined.

$$\int_{-1}^{0} \frac{1}{\sqrt[3]{x^2}} dx + \int_{0}^{2} \frac{1}{\sqrt[3]{x^2}} dx$$

$$\lim_{b\to 0} \int_{-1}^{b} \frac{1}{\sqrt[3]{x^2}} dx + \lim_{a\to 0} \int_{a}^{2} \frac{1}{\sqrt[3]{x^2}} dx = \lim_{b\to 0} \int_{-1}^{b} x^{-\frac{2}{3}} dx + \lim_{a\to 0} \int_{a}^{2} x^{-\frac{2}{3}} dx =$$

$$\lim_{b\to 0} \left[3x^{\frac{1}{3}}\right]_{-1}^{b} + \lim_{a\to 0} \left[3x^{\frac{1}{3}}\right]_{a}^{2} = \lim_{b\to 0}\left[3\sqrt[3]{b} - 3\sqrt[3]{-1}\right] + \lim_{a\to 0}\left[3\sqrt[3]{2} - 3\sqrt[3]{a}\right] =$$

$$3 + 3\sqrt[3]{2}$$

This also converges to the above value.

Example.

$$\int_{0}^{\infty} \frac{1}{x} dx$$

$$\lim_{a\to 0} \int_{a}^{1} \frac{1}{x} dx + \lim_{b\to\infty} \int_{1}^{b} \frac{1}{x} dx = \lim_{a\to 0} [\ln x]_a^1 + \lim_{b\to\infty} [\ln x]_1^b =$$

$$\lim_{a\to 0}[\ln 1 - \ln a] + \lim_{b\to\infty} [\ln b - \ln 1] = \lim_{a\to 0}[-\ln a] + \lim_{b\to\infty} [\ln b] = -(-\infty) + \infty = \infty$$

The integral diverges.

Exam #1
Covers Chapters 2 and 3 from the online textbook:
Applications: area between curves, volumes and surface areas of revolution, arc length, work, center of mass, growth/decay, hyperbolic trig functions
Integration Techniques: integration by part, trig integrals, trig substitution, partial fractions, numerical integration, improper integrals (only use tables of integrals if specifically directed to).