3/26/2024

Taylor Series/Taylor Polynomials (6.3)

Taylor series are an infinite power series that is obtained from taking the derivative(s) of the function to represented. A Taylor polynomial is any finite subset (finite series) of the infinite series that approximates the function “near” the center. The Maclaurin series is a Taylor series but where the center is at x=0, while a Taylor series can be centered at any value of x in the domain.

The general formula for Taylor series:

$$f(x) \approx f(c) + f'(c)(x-c) + \frac{f''(c)(x-c)^2}{2} + \frac{f'''(c)(x-c)^3}{6} + \frac{f^{IV}(c)(x-c)^4}{4!} + \cdots \frac{f^{(n)}(c)(x-c)^n}{n!}$$

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(c)}{n!}(x-c)^n$$

The kinds of functions we should generally apply this to are functions that are not rational functions (like $e^x, \sin(x), \cos(x), \tan(x)$ ...) and not things like ($\frac{1}{x}, \frac{x}{x^2-1} = -\frac{x}{1-x^2}, etc.$).

Example.
Find the Maclaurin polynomial for the function $f(x) = e^x$ for n=5, and then write a general formula for the Maclaurin series.

n	$n!$	$f^{(n)}(x)$	$f^{(n)}(0)$	$(x-0)^n = x^n$	$\frac{f^{(n)}(c)}{n!}(x-c)^n$
0	1	e^x	1	1	$\frac{1}{1}(1) = 1$
1	1	e^x	1	x	$\frac{1}{1}(x) = x$
2	2	e^x	1	x^2	$\frac{1}{2}(x^2) = \frac{x^2}{2}$
3	6	e^x	1	x^3	$\frac{1}{6}(x^3) = \frac{x^3}{6}$
4	24	e^x	1	x^4	$\frac{1}{24}(x^4) = \frac{x^4}{24}$
5	120	e^x	1	x^5	$\frac{1}{120}(x^5) = \frac{x^5}{120}$

$$P_5(x) = 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24} + \frac{x^5}{120}$$

$$f(x) = \sum_{n=0}^{\infty} \frac{x^n}{n!}$$

Example.

Find the Taylor series for the function $f(x) = \sqrt{x}$, centered at $c = 4$. Find the polynomial to $n = 6$, and the general formula for the Taylor series.

n	$n!$	$f^{(n)}(x)$	$f^{(n)}(4)$	$(x-4)^n$	$\frac{f^{(n)}(c)}{n!}(x-c)^n$
0	1	$x^{\frac{1}{2}}$	$2 = 2^1$	1	$\frac{(2)}{1}(1) = 2$
1	1	$\frac{1}{2}x^{-1/2}$	$\frac{1}{2}\left(\frac{1}{2} = 2^{-1}\right) = \frac{1}{4}$	$(x-4)$	$\frac{\frac{1}{4}}{1}(x-4) = \frac{1}{4}(x-4)$
2	2	$-\frac{1}{4}x^{-\frac{3}{2}}$	$\left(-\frac{1}{4}\right)\left(\frac{1}{8} = 2^{-3}\right) = -\frac{1}{32} = -\frac{1}{2^5}$	$(x-4)^2$	$-\frac{\frac{1}{32}}{2}(x-4)^2 = -\frac{1}{64}(x-4)^2$
3	6	$\frac{3}{8}x^{-\frac{5}{2}}$	$\left(\frac{3}{8}\right)\left(\frac{1}{32} = 2^{-5}\right) = \frac{3}{256} = \frac{3}{2^7}$	$(x-4)^3$	$\frac{\frac{3}{256}}{6}(x-4)^3 = \frac{1}{512}(x-4)^3$
4	24	$-\frac{15}{16}x^{-\frac{7}{2}}$	$\left(-\frac{15}{16}\right)\left(\frac{1}{128} = 2^{-7}\right) = -\frac{15}{2048} = -\frac{15}{2^{11}}$	$(x-4)^4$	$-\frac{\frac{15}{2048}}{24}(x-4)^4 = -\frac{5}{16384}(x-4)^4$
5	120	$\frac{105}{32}x^{-\frac{9}{2}}$	$\left(\frac{105}{32}\right)\left(\frac{1}{512} = 2^{-9}\right) = \frac{105}{16384} = \frac{105}{2^{14}}$	$(x-4)^5$	$\frac{\frac{105}{16384}}{120}(x-4)^5 = \frac{7}{131072}(x-4)^5$
6	720	$-\frac{945}{64}x^{-\frac{11}{2}}$	$\left(-\frac{945}{64}\right)\left(\frac{1}{2048} = 2^{-11}\right) = -\frac{945}{131072} = -\frac{945}{2^{17}}$	$(x-4)^6$	$-\frac{\frac{945}{131072}}{720}(x-4)^6 = \frac{-21}{2097152}(x-4)^6$
n	$n!$	$\frac{(-1)^n(2n+1)(2n-1)(2n-3)\ldots}{2^n}$		$(x-4)^n$	

$$P_6 = 2 + \frac{1}{4}(x-4) - \frac{1}{64}(x-4)^2 + \frac{1}{512}(x-4)^3 - \frac{5}{16384}(x-4)^4 + \frac{7}{131072}(x-4)^5 - \frac{21}{2097152}(x-4)^6$$

$$f(x) = 2 + \sum_{n=2}^{\infty} \frac{(-1)^n [1 \cdots (2n-7)(2n-5)(2n-3)](x-4)^n}{n!\, 2^n (2^{1-2n})}$$

Example.

Find the Taylor polynomial with $n = 8$ centered at $x = 0$, for the function $\cos(x) = f(x)$.

n	$n!$	$f^{(n)}(x)$	$f^{(n)}(0)$	$(x-0)^n = x^n$	$\frac{f^{(n)}(c)}{n!}(x-c)^n$
0	1	$\cos(x)$	1	1	1 (k=0)
1	1	$-\sin(x)$	0	x	0
2	2	$-\cos(x)$	-1	x^2	$-\frac{x^2}{2}$ (k=1)
3	6	$\sin(x)$	0	x^3	0
4	24	$\cos(x)$	1	x^4	$\frac{x^4}{24}$ (k=2)
5	120	$-\sin(x)$	0	x^5	0
6	720	$-\cos(x)$	-1	x^6	$-\frac{x^6}{720}$ (k=3)
7	5040	$\sin(x)$	0	x^7	0
8	40320	$\cos(x)$	1	x^8	$\frac{x^8}{8!}$ (k=4)

$$P_8 = 1 - \frac{x^2}{2} + \frac{x^4}{24} - \frac{x^6}{720} + \frac{x^8}{40320}$$

$$f(x) = \sum_{k=0}^{\infty} (-1)^k \frac{x^{2k}}{(2k)!}$$

Taylor Series Remainder (Error Term)

$$|R_n(x)| \leq \frac{\max\left|f^{(n+1)}(x)\right| (x-c)^{n+1}}{(n+1)!}$$

For all intents and purposes, somewhat similar to the alternating series test, this expression is based on the next term in the series after we stop.

The function is not evaluated at the given point x, but the maximum on some interval is used in the estimate.

That interval could be the center to the point in question, it could be specified in the problem, sometimes it's the symmetric interval [-a,a], where a is the point we are evaluating the error at.

Example.

Using the Taylor polynomial for $f(x) = \cos(x)$ derived above, what is the error on $P_8\left(\frac{\pi}{8}\right)$?

$$|R_n(x)| \leq \frac{\max\left|f^{(n+1)}(x)\right|(x-c)^{n+1}}{(n+1)!}$$

$$\left|R_8\left(\frac{\pi}{8}\right)\right| \leq \frac{(1)\left(\frac{\pi}{8}-0\right)^9}{(9)!} \approx 6.12034 \ldots \times 10^{-10}$$

What is the error estimate at π?

$$|R_8(\pi)| \leq \frac{(1)(\pi-0)^9}{(9)!} \approx 0.0821 \ldots$$

Example.
What is the error estimate on e^1 using the P_5 Taylor polynomial for $f(x) = e^x$.

Intervals that would be reasonable to use: $[-1,1], [0,1]$
The intervals would have to include the center, and the point being evaluated/estimated

This function is increasing, so the maximum will be at the largest value of x.
If the function is decreasing, then the maximum will be at the smallest value of x.
If it changes direction, then look for a critical point inside the interval.

$$|R_n(x)| \leq \frac{\max\left|f^{(n+1)}(x)\right|(x-c)^{n+1}}{(n+1)!}$$

$$|R_5(1)| \leq \frac{e^1(1-0)^6}{(6)!} \approx 0.00377 \ldots$$

Can also use this remainder to determine the number of terms needed to estimate a function at particular value to a certain tolerance.

On Thursday:
Operations on Taylor series:
Combine them through addition/subtraction, multiplication, etc.
Composition with Taylor series
Limits and integration with Taylor series