2/27/2025

Inverse Trig Functions

Use inverse trig functions when we have the value of the trig function (one of the six) and we want to determine what angle that value goes with (belongs to).

All of our trig functions fail the horizontal line test: i.e. not one-to-one.

Our inverses will be relations (for the entire function), therefore the inverse is not a function.
To make the inverses into functions, we restrict the domain of the original function so that it is one-to-one.

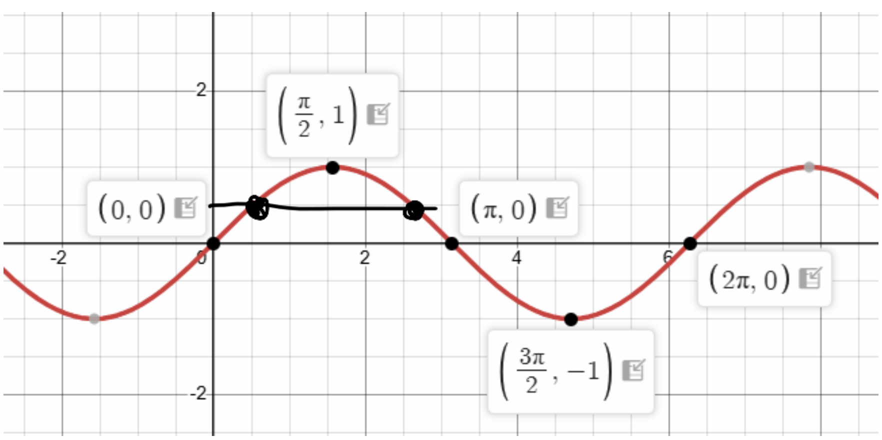
We also prefer the domains and ranges to be continuous. We want to cover everything in the range (minimum to maximum) and do it all in one piece.

Notation:
Inverse function notation uses a $f^{-1}(x)$ notation to indicate an inverse. $f^{-1}(x) \neq \frac{1}{f(x)}$.

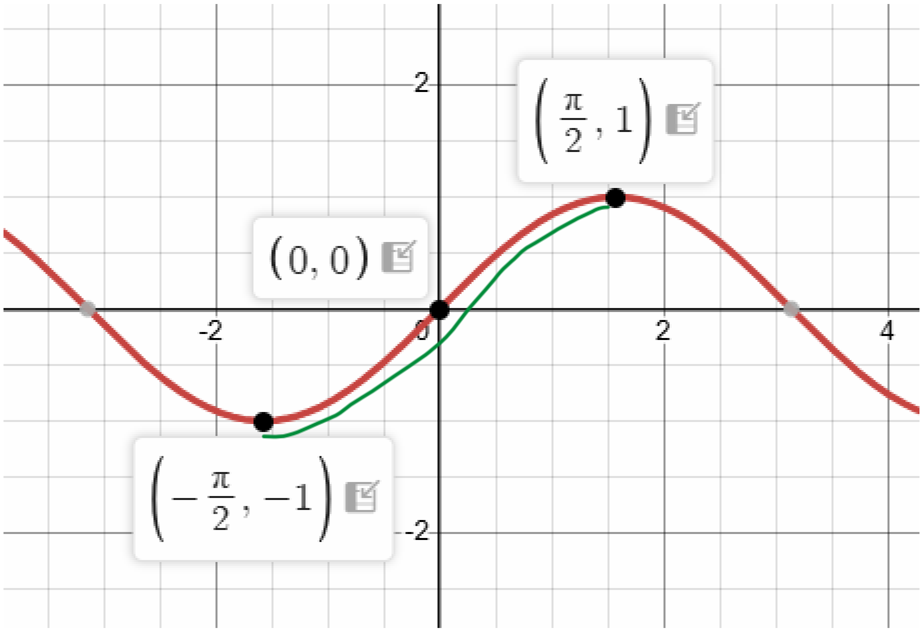
$$\sin^{-1}(x) \neq \frac{1}{\sin(x)}$$
$$(\sin(x))^{-1} = \frac{1}{\sin(x)} = \csc x$$

$$\sin^{-1}(x) = \arcsin(x) = \operatorname{asin}(x)$$

Defining the Inverse Sine Function.



Restrict the domain:

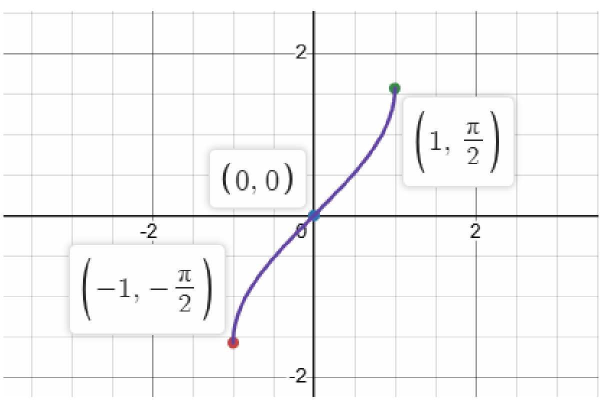


This section of graph goes from the minimum to the maximum and includes 0, is all in one piece.

Domain is restricted to $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$. Range $[-1,1]$

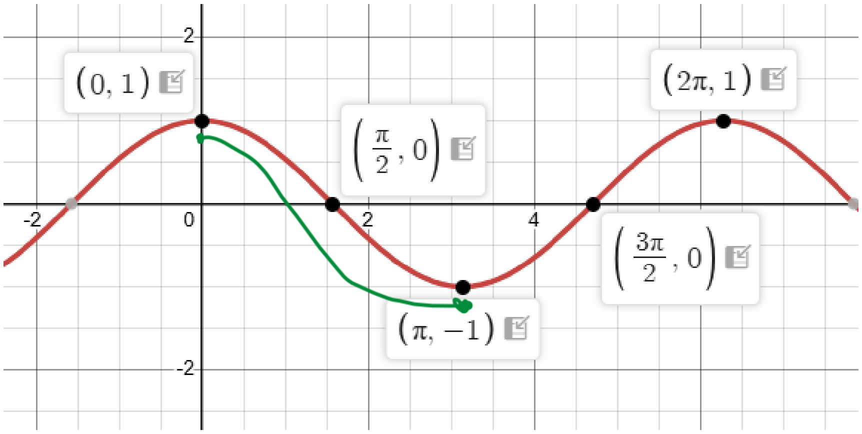
Finding the inverse: we take every point and switch the x and y coordinates.
For our inverse sine function, the domain is $[-1,1]$, and the range is $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$.

From the original sine function: $(0,0), \left(\frac{\pi}{2}, 1\right), \left(-\frac{\pi}{2}, -1\right)$
So, on the inverse we'll have points: $\left(-1, -\frac{\pi}{2}\right), (0,0), \left(1, \frac{\pi}{2}\right)$.



Graph of the arcsin(x)
The function takes a value of sine and converts it back into an angle.

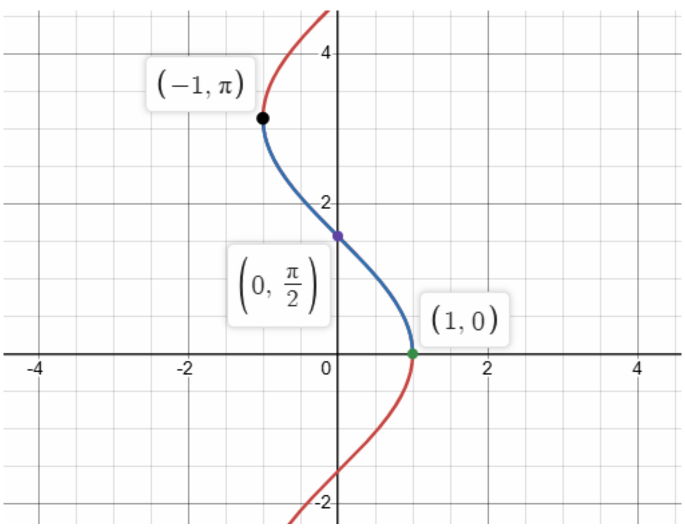
Inverse Cosine Function



The original function is not one-to-one, so we restrict the domain to $[0, \pi]$, the range is still $[-1,1]$.

Key points on the original function: $(0,1), \left(\frac{\pi}{2}, 0\right), (\pi, -1)$
On the inverse: $(1,0), \left(0, \frac{\pi}{2}\right), (-1, \pi)$

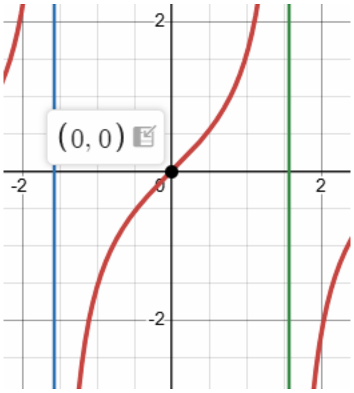
$$\cos^{-1}(x) = \arccos(x)$$



Domain of arccosine is $[-1,1]$, range $[0, \pi]$

The inverse sine produces angles in the first and fourth quadrants, while inverse cosine produces angles in the first and second quadrants.

Inverse Tangent Function

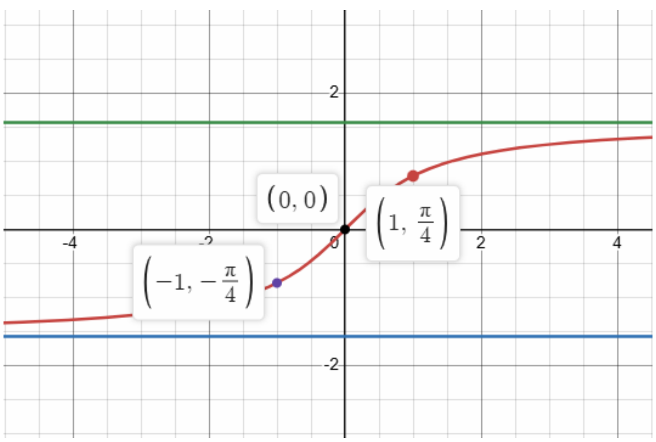


Original tangent function have vertical asymptotes at $x = -\frac{\pi}{2}, x = \frac{\pi}{2}$, keypoints $\left(-\frac{\pi}{4}, -1\right), (0,0), \left(\frac{\pi}{4}, 1\right)$

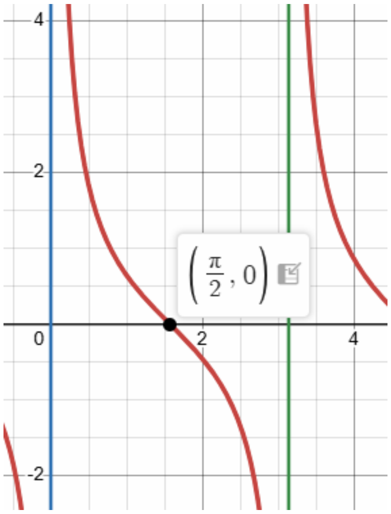
Restrict the domain to be between the two asymptotes closest to 0 (i.e. $-\frac{\pi}{2}, \frac{\pi}{2}$).
Domain: $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$, range is $(-\infty, \infty)$

When we create the inverse function, we have:
Horizontal asymptotes: $y = -\frac{\pi}{2}, y = \frac{\pi}{2}$
Keypoints: $\left(-1, -\frac{\pi}{4}\right), (0,0), \left(1, \frac{\pi}{4}\right)$

Domain of inverse tangent function: $(-\infty, \infty)$, the range is $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$



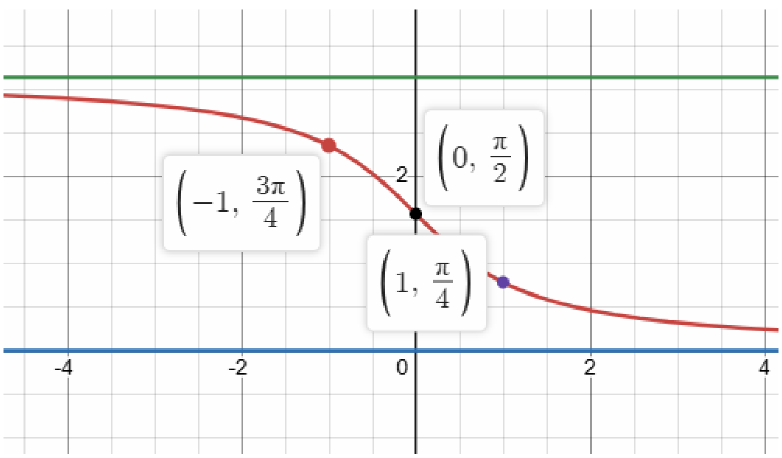
Inverse Cotangent Function



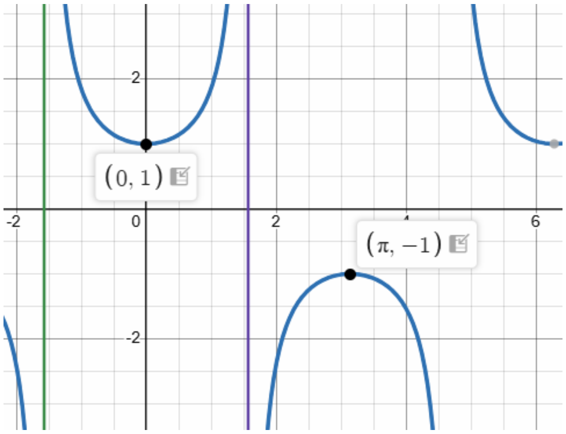
Original keypoints: vertical asymptote at $x = 0, x = \pi$, $\left(\frac{\pi}{4}, 1\right), \left(\frac{\pi}{2}, 0\right) \left(\frac{3\pi}{4}, -1\right)$

Restrict the domain to between our two asymptotes $(0, \pi)$

Inverse Cotangent Function:
Domain: $(-\infty, \infty)$, Range: $(0, \pi)$
Keypoints: horizontal asymptote at y=0, $y = \pi$, $\left(1, \frac{\pi}{4}\right), \left(0, \frac{\pi}{2}\right), \left(\frac{3\pi}{4}, -1\right)$

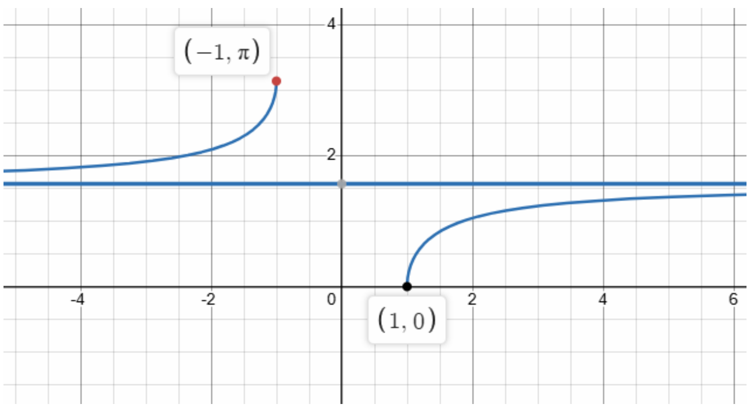


Inverse Secant

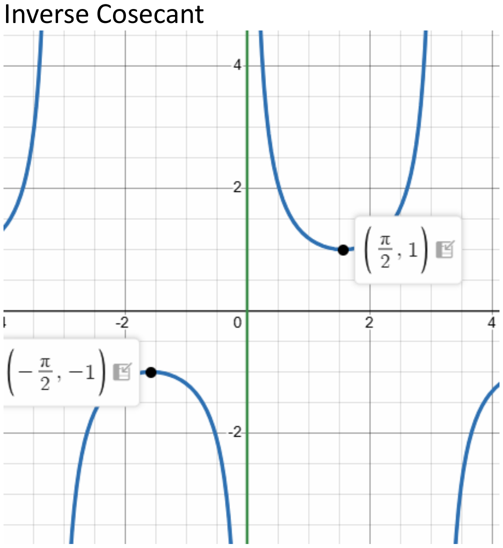


The restricted domain will go from $\left[0, \frac{\pi}{2}\right) \cup \left(\frac{\pi}{2}, \pi\right]$
Range: $(-\infty, -1] \cup [1, \infty)$

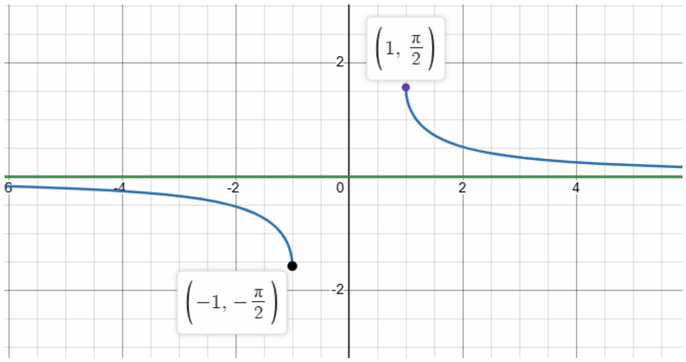
Inverse Domain is $(-\infty, -1] \cup [1, \infty)$, inverse range: $\left[0, \frac{\pi}{2}\right) \cup \left(\frac{\pi}{2}, \pi\right]$



Inverse Cosecant



Domain restricted to $\left[-\frac{\pi}{2}, 0\right) \cup \left(0, \frac{\pi}{2}\right]$, range $(-\infty, -1] \cup [1, \infty)$
Inverse domain is $(-\infty, -1] \cup [1, \infty)$, and inverse range is $\left[-\frac{\pi}{2}, 0\right) \cup \left(0, \frac{\pi}{2}\right]$



$$\cot(x) = \frac{1}{\tan(x)}, \sec(x) = \frac{1}{\cos(x)}, \csc(x) = \frac{1}{\sin(x)}$$

$$\tan(x) = \frac{1}{\cot(x)}, \cos(x) = \frac{1}{\sec(x)}, \sin(x) = \frac{1}{\csc(x)}$$

$$\cot(a) = \frac{4}{3}$$
$$\tan(a) = \frac{3}{4}$$
$$\arctan\left(\frac{3}{4}\right) = a$$

$$\operatorname{arccot}\left(\frac{4}{3}\right) = \arctan\left(\frac{3}{4}\right) = a$$

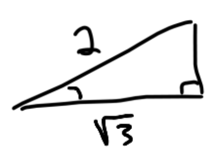
$$\operatorname{arcsec}(4) = \arccos\left(\frac{1}{4}\right)$$
$$\operatorname{arccsc}(3) = \arcsin\left(\frac{1}{3}\right)$$

Inverse cosine and inverse secant have the same domain except for $x = \frac{\pi}{2}$.
Inverse sine and inverse cosecant have the same domain except for x=0

Inverse tangent has a domain of first and fourth quadrants, while the inverse cotangent first and second quadrants. They overlap in the first quadrant. If the value of cotangent is positive, then the inverse tangent will give you the correct angle. However, if your value is negative, you will get an angle in the fourth quadrant and not in the second quadrant where it should be.

Evaluate

$$\sin^{-1}\left(\frac{1}{2}\right) = \theta = \frac{\pi}{6}$$
$$\sin(\theta) = \frac{1}{2}$$



Evaluate

$$\cos^{-1}\left(\cos\frac{4\pi}{3}\right) = \cos^{-1}\left(-\frac{\sqrt{3}}{2}\right) = \frac{2\pi}{3}$$

Compare with

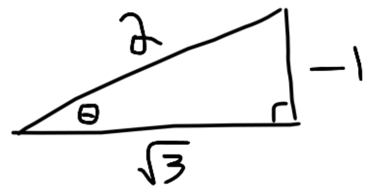
$$\cos^{-1}\left(\cos\frac{\pi}{3}\right) = \frac{\pi}{3}$$

The inverse and the original function will cancel, but only inside the restricted domain.

$$\sin\left(\sin^{-1}\pi\right) = undefined$$

Simplify

$$\sec\left(\sin^{-1} -\frac{1}{2}\right)$$

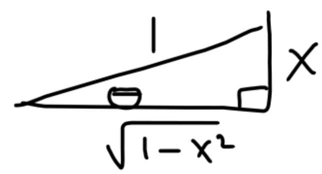


$$\sin(\theta) = -\frac{1}{2}$$

The domain of the inverse sine is in the first and fourth quadrants.

$$\theta = -\frac{\pi}{6}$$

$$\sec\left(-\frac{\pi}{6}\right) = \sec(\theta) = \frac{2}{\sqrt{3}}$$



$$\sec(\sin^{-1} x)$$

Pick with applying transformation to inverse trig functions.