2/6/2025

Transformations
Angles and Radian Measure
Trigonometric Functions (unit circle)

If f and g are inverses of each other, then $f\big(g(x)\big) = g\big(f(x)\big) = x$

$$(f \circ g)(x) = (g \circ f)(x)$$

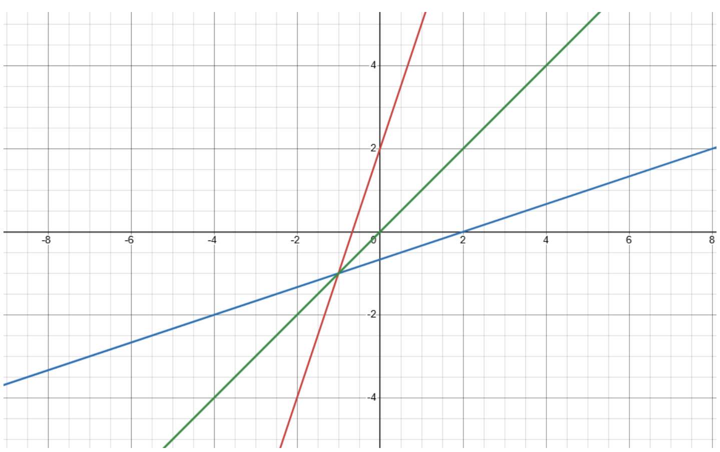
Consider the function $f(x) = 3x + 2$
Find the inverse:

$$y = 3x + 2$$
$$x = 3y + 2$$
$$x - 2 = 3y$$
$$\frac{x-2}{3} = y$$

$$f^{-1}(x) = \frac{x-2}{3}$$

$$f\big(f^{-1}(x)\big) = 3\left(\frac{x-2}{3}\right) + 2 = x - 2 + 2 = x$$

$$f^{-1}\big(f(x)\big) = \frac{(3x+2)-2}{3} = \frac{3x+2-2}{3} = \frac{3x}{3} = x$$



Transformations
Horizontal transformations:

- Horizontal shift (moving the graph left or right)
- Horizontal stretch/compression (you stretching or compressing horizontally)
- Horizontal reflection (moving the right side to the left side, and left to right)

Vertical transformations:

- Vertical shift (moving it up or down)

- Vertical stretching or compressing (stretching or compressing up and down)
- Vertical reflections (bottom to the top, and the top to the bottom)

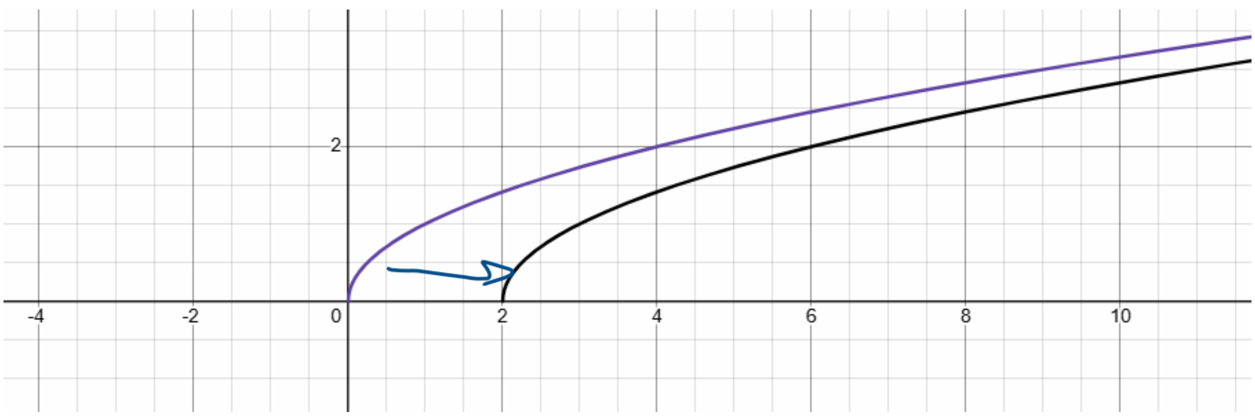
Base functions/library functions

Identity function $y = x$, D: $(-\infty, \infty)$, R: $(-\infty, \infty)$
Square function: $y = x^2$, D: $(-\infty, \infty)$, R: $[0, \infty)$
Cube function: $y = x^3$, D: $(-\infty, \infty)$, R: $(-\infty, \infty)$
Square root function: $y = \sqrt{x}$, D: $[0, \infty)$, R: $[0, \infty)$
Cube root function: $y = \sqrt[3]{x}$, D: $(-\infty, \infty)$, R: $(-\infty, \infty)$
Absolute value function: $y = |x|$, D: $(-\infty, \infty)$, R: $[0, \infty)$
Exponential function: $y = e^x$, D:$(-\infty, \infty)$, R: $(0, \infty)$
Natural log function: $y = \ln(x)$, D: $(0, \infty)$, R: $(-\infty, \infty)$

Base function is $f(x) = \sqrt{x}$

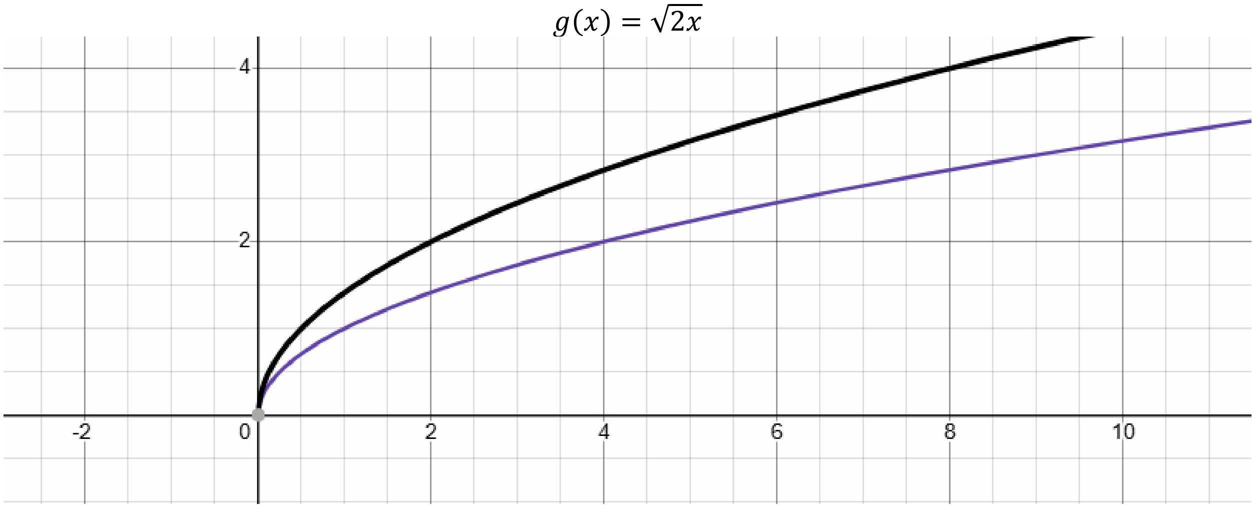
Apply a horizontal shift of 2 to the right: $f(x-2) = g(x)$

$$g(x) = \sqrt{x-2}$$



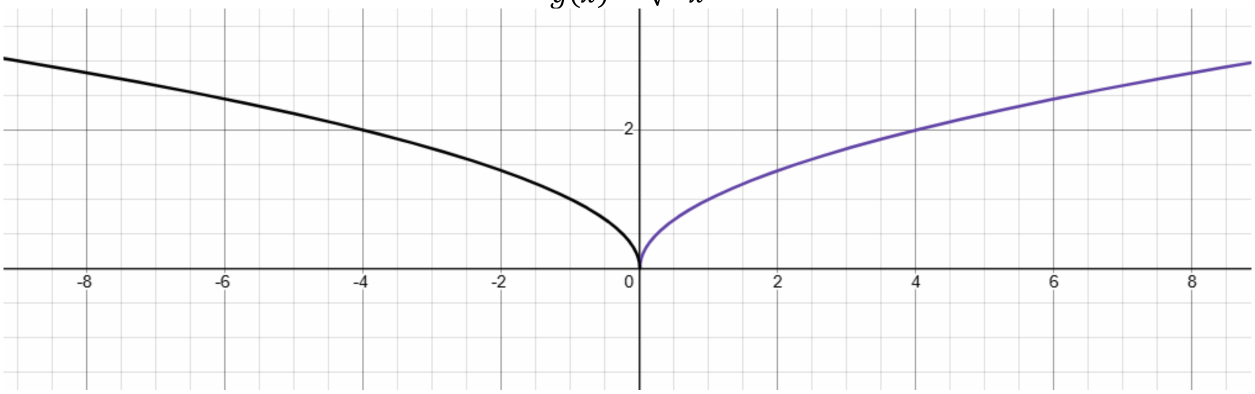
Horizontal compression: $f(ax), a > 1$, stretch : $f(ax), 0 < a < 1$

$$g(x) = \sqrt{2x}$$



Horizontal reflections : $f(-x)$: replace x with -x

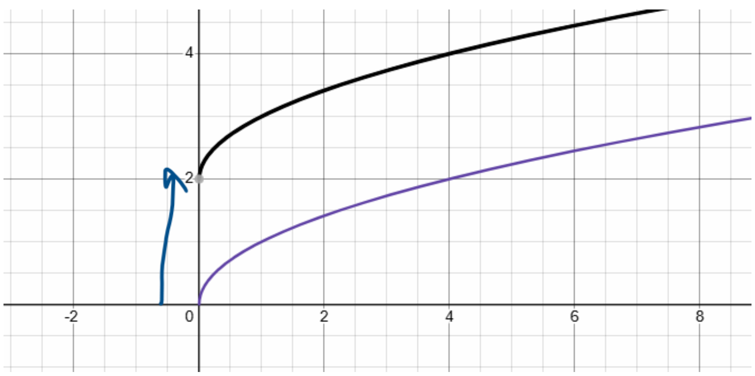
$$g(x) = \sqrt{-x}$$



Vertical shift: $f(x) + k = g(x)$

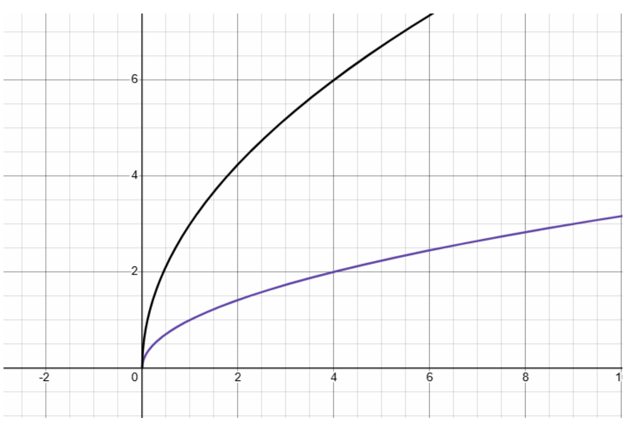
$$g(x) = \sqrt{x} + 2$$

Upward shift of 2 units



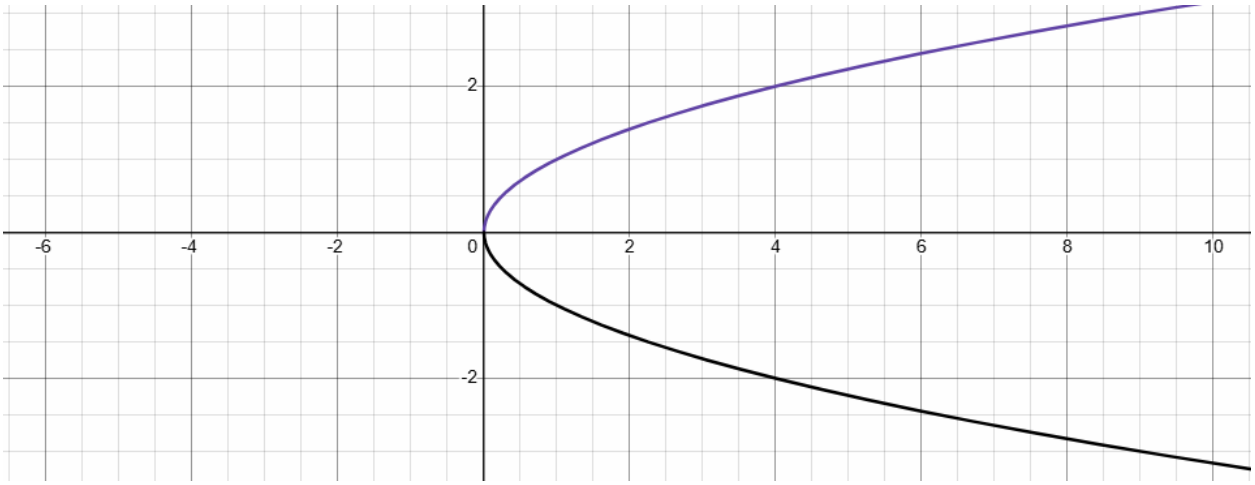
Vertical Stretch: $af(x), a > 1$, vertical compression: $af(x), 0 < a < 1$,

$$g(x) = 3\sqrt{x}$$



Vertical reflection: $-f(x)$

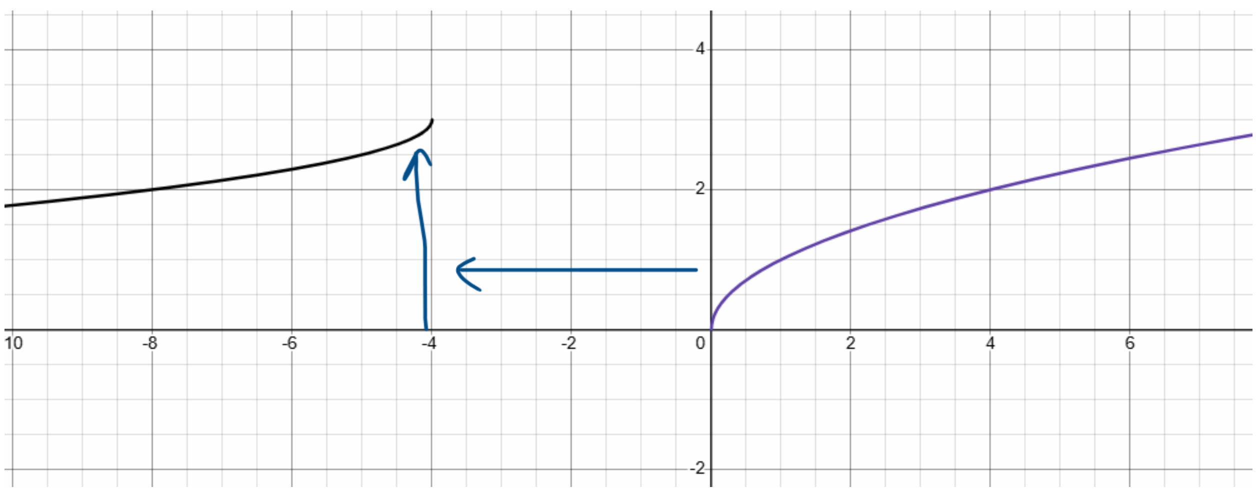
$$g(x) = -\sqrt{x}$$



Typically you apply horizontal transformations first (in the order reflection/compression, then shift), and then any vertical transformations (in order reflection/compression, then shift).

$$g(x) = -\frac{1}{2}\sqrt{-(x+4)} + 3$$

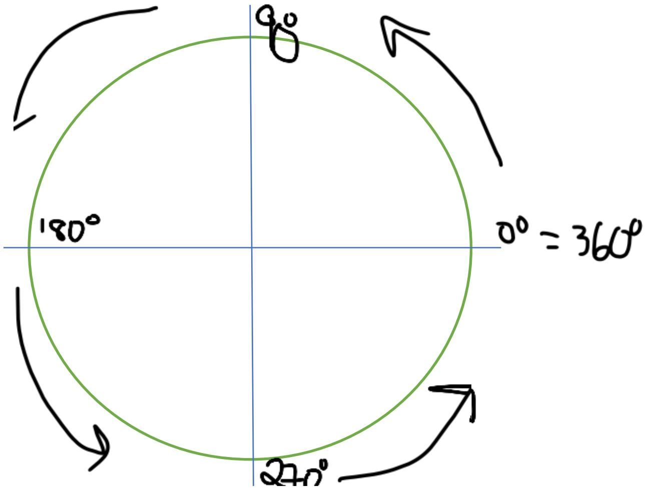
Vertical reflection (negative out front)
Vertical compression (1/2 out front)
Vertical shift (+3 at the end, outside the function)
Horizontal reflection (- in front of the x (under the root))
Horizontal shift (left by 4)



Domain of g: $(-\infty, -4]$, range of g: $(-\infty, 3]$

Angles
In a full circle, there are 360°. 180-degrees is a straight line, 90-degrees is a right angle, and half of a right-angle is 45-degrees. Acute angles are angles that are less than 90-degrees. Obtuse angles are angles that are between 90 and 180-degrees. Isosceles right-triangle: 45-45-90 triangle. Equilateral triangle (divided in half): 60-60-60 (30-60-90)

In mathematics, the positive x-axis is considered 0-degrees, and angles increase as you go counterclockwise.



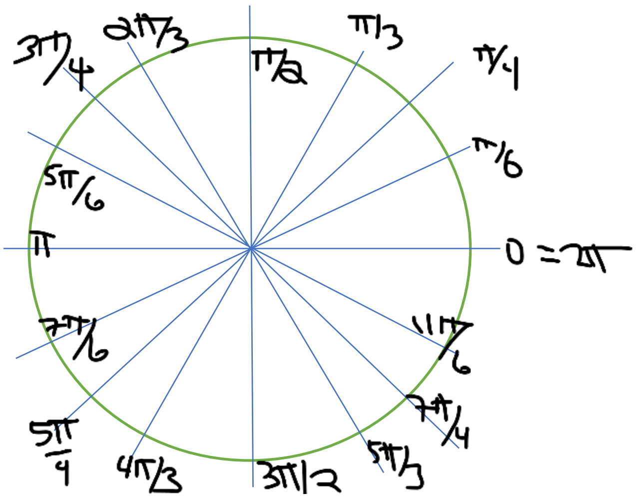
Radians: every circle has 2π radians in the circle

Conversion from degrees to radians is $\frac{\pi}{180°}$, conversion from radians to degrees is $\frac{180°}{\pi}$.

225-degrees, what is this in radians. $225°\left(\frac{\pi}{180°}\right) = \frac{5}{4}\pi$

What is $\frac{7\pi}{8}$ radians as degrees? $\frac{7\pi}{8}\left(\frac{180°}{\pi}\right) = 157.5°$

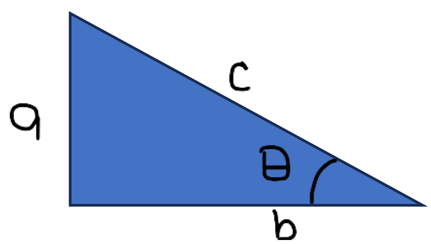
$$0° = 0\ radians$$
$$30° = \frac{\pi}{6}\ radians$$
$$45° = \frac{\pi}{4}\ radians$$
$$60° = \frac{\pi}{3}\ radians$$
$$90° = \frac{\pi}{2}\ radians$$



Length of an arc: if a circle has a radius of r and we pass around the circumference through an angle of 120°, what is the length of the arc travelled?

Similarly for area of a sector: angle must be expressed in radians for the formula to work.

Defining Trig functions from right triangles:



Angle θ

$$\sin\theta = \frac{opposite}{hypotenuse} = \frac{a}{c}$$

$$\cos\theta = \frac{adjacent}{hypotenuse} = \frac{b}{c}$$

$$\tan\theta = \frac{opposite}{adjacent} = \frac{a}{b}$$

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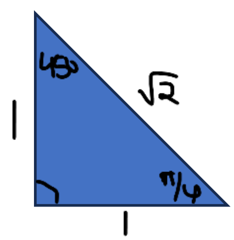
$$\cot\theta = \frac{1}{\tan\theta} = \frac{adjacent}{opposite} = \frac{b}{a}$$

$$\sec\theta = \frac{1}{\cos\theta} = \frac{hypotenuse}{adjacent} = \frac{c}{b}$$

$$\csc\theta = \frac{1}{\sin\theta} = \frac{hypotenuse}{opposite} = \frac{c}{a}$$



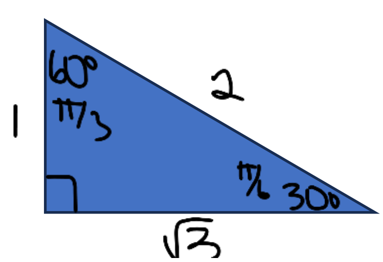
45-45-90

$$\sin 45° = \sin\frac{\pi}{4} = \frac{1}{\sqrt{2}}$$
$$\cos 45° = \cos\frac{\pi}{4} = \frac{1}{\sqrt{2}}$$

$$\tan 45° = \tan\frac{\pi}{4} = \frac{1}{1} = 1$$

$$\cot 45° = \cot\frac{\pi}{4} = 1$$
$$\sec 45° = \sec\frac{\pi}{4} = \sqrt{2}$$

$$\csc 45° = \csc\frac{\pi}{4} = \sqrt{2}$$



$$\sin 30° = \sin\frac{\pi}{6} = \frac{1}{2}$$
$$\cos 30° = \cos\frac{\pi}{6} = \frac{\sqrt{3}}{2}$$

$$\tan 30° = \tan\frac{\pi}{6} = \frac{1}{\sqrt{3}}$$

$$\cot 30° = \cot\frac{\pi}{6} = \sqrt{3}$$
$$\sec 30° = \sec\frac{\pi}{6} = \frac{2}{\sqrt{3}}$$

$$\csc 30° = \csc\frac{\pi}{6} = 2$$

$$\sin 60° = \sin\frac{\pi}{3} = \frac{\sqrt{3}}{2}$$
$$\cos 60° = \cos\frac{\pi}{3} = \frac{1}{2}$$

$$\tan 60° = \tan\frac{\pi}{3} = \sqrt{3}$$

$$\cot 60° = \cot\frac{\pi}{3} = \frac{1}{\sqrt{3}}$$
$$\sec 60° = \sec\frac{\pi}{3} = 2$$

$$\csc 60° = \csc\frac{\pi}{3} = \frac{2}{\sqrt{3}}$$

One thing to note here:

$$\sin 60° = \sin\frac{\pi}{3} = \frac{\sqrt{3}}{2}$$

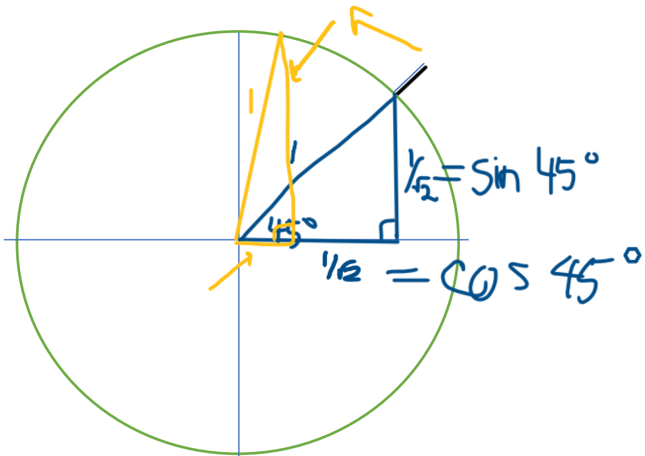
$$\cos 30° = \cos\frac{\pi}{6} = \frac{\sqrt{3}}{2}$$

These are complementary angles (the angles add to 90-degrees), and the sine is equal to the cosine.

$$\cos 60° = \cos\frac{\pi}{3} = \frac{1}{2}$$
$$\sin 30° = \sin\frac{\pi}{6} = \frac{1}{2}$$

Our reference angles for angles and trig functions in other quadrants.

How do our trig functions in acute case relate to the unit circle?



$$x^2 + y^2 = 1$$

As the angle approaches 90-degrees, the sine value goes to 1 (because the opposite side gets closer in size the hypotenuse).
In the same case, the cosine value will get closer to 0 because the adjacent side is getting shorter.

When the angle gets closer to 0-degrees, the vertical side (sine value) is getting shorter and closer to 0, and the cosine value is getting larger and closer to the size of the radius (closer to 1).

$$\sin 90° = \sin\frac{\pi}{2} = 1$$
$$\cos 90° = \cos\frac{\pi}{2} = 0$$
$$\sin 0° = \sin 0 = 0$$
$$\cos 0° = \cos 0 = 1$$

Next time, we will extend the trig functions into quadrants 2-4.