3/11/2025

Sum and Difference Formulas
Product-to-Sum Formulas

Sum and Difference formulas:

$$\sin(\alpha+\beta)=\sin\alpha\cos\beta+\cos\alpha\sin\beta$$
$$\sin(\alpha-\beta)=\sin\alpha\cos\beta-\cos\alpha\sin\beta$$
$$\cos(\alpha+\beta)=\cos\alpha\cos\beta+\sin\alpha\sin\beta$$
$$\cos(\alpha-\beta)=\cos\alpha\cos\beta-\sin\alpha\sin\beta$$
$$\tan(\alpha+\beta)=\frac{\tan\alpha+\tan\beta}{1-\tan\alpha\tan\beta}$$
$$\tan(\alpha-\beta)=\frac{\tan\alpha-\tan\beta}{1+\tan\alpha\tan\beta}$$

Example.

$$\cos(75°)=\cos(45°+30°)=$$
$$\cos(45°)\cos(30°)+\sin(45°)\sin(30°)=\left(\frac{1}{\sqrt{2}}\right)\left(\frac{\sqrt{3}}{2}\right)+\left(\frac{1}{\sqrt{2}}\right)\left(\frac{1}{2}\right)=\frac{\sqrt{3}+1}{2\sqrt{2}}$$

$$\sec(75°)=\frac{2\sqrt{2}}{\sqrt{3}+1}$$

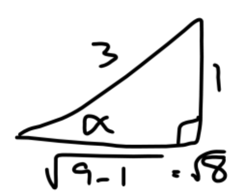
Example.

$$\tan\left(\frac{13\pi}{12}\right)=\tan\left(\frac{\pi}{4}+\frac{5\pi}{6}\right)=\tan\left(\frac{3\pi}{4}+\frac{\pi}{3}\right)$$

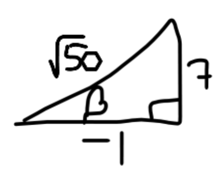
$$=\frac{\tan\left(\frac{\pi}{4}\right)+\tan\left(\frac{5\pi}{6}\right)}{1-\tan\left(\frac{\pi}{4}\right)\tan\left(\frac{5\pi}{6}\right)}=\frac{1-\frac{1}{\sqrt{3}}}{1-(1)\left(-\frac{1}{\sqrt{3}}\right)}=\frac{1-\frac{1}{\sqrt{3}}}{1+\frac{1}{\sqrt{3}}}\left(\frac{\sqrt{3}}{\sqrt{3}}\right)=\frac{\sqrt{3}-1}{\sqrt{3}+1}$$

Example.

Suppose $\csc\alpha=3$, in Q1 $\left[0,\frac{\pi}{2}\right]$, $\tan\beta=-7$, Q2 $\left[\frac{\pi}{2},\pi\right]$.
Find the following angles:



$$\sin(\alpha-\beta)=\sin\alpha\cos\beta-\cos\alpha\sin\beta$$
$$\left(\frac{1}{3}\right)\left(-\frac{1}{\sqrt{50}}\right)-\left(\frac{\sqrt{8}}{3}\right)\left(\frac{7}{\sqrt{50}}\right)=\frac{-1-7\sqrt{8}}{3\sqrt{50}}$$



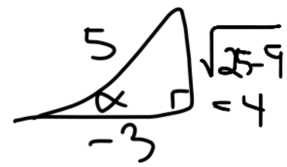
$$\tan(\alpha+\beta)=\frac{\tan\alpha+\tan\beta}{1-\tan\alpha\tan\beta}=$$

$$\frac{\frac{1}{\sqrt{8}}-7}{1-\left(\frac{1}{\sqrt{8}}\right)(-7)}\left(\frac{\sqrt{8}}{\sqrt{8}}\right)=\frac{(1-7\sqrt{8})}{\sqrt{8}+7}$$

Example.

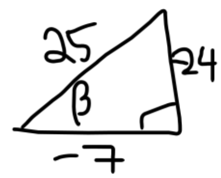
Suppose $\sec(\alpha)=-\frac{5}{3}$, Q2, and $\tan\beta=\frac{24}{7}$, in Q3

Find the following values:



$$\csc(\alpha-\beta)=\frac{1}{\sin(\alpha-\beta)}=-\frac{5}{4}$$

$$\sin(\alpha-\beta)=\sin\alpha\cos\beta-\cos\alpha\sin\beta$$
$$=\left(\frac{4}{5}\right)\left(-\frac{7}{25}\right)-\left(-\frac{3}{5}\right)\left(-\frac{24}{25}\right)=\frac{-28-72}{125}=-\frac{100}{125}=-\frac{4}{5}$$



$$\cot(\alpha-\beta)=\frac{1}{\tan(\alpha-\beta)}=\frac{1+\tan\alpha\tan\beta}{\tan\alpha-\tan\beta}$$

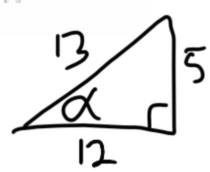
$$\tan(\alpha-\beta)=\frac{\tan\alpha-\tan\beta}{1+\tan\alpha\tan\beta}$$

$$\frac{1+\tan\alpha\tan\beta}{\tan\alpha-\tan\beta}=\frac{1+\left(\frac{4}{-3}\right)\left(\frac{24}{7}\right)}{\left(-\frac{4}{3}\right)-\left(\frac{24}{7}\right)}=\frac{1-\frac{32}{7}}{-\frac{100}{21}}\left(\frac{21}{21}\right)=\frac{21-96}{-100}=-\frac{75}{-100}=\frac{3}{4}$$

Example.

155. $\sin\left(\arcsin\left(\frac{5}{13}\right)+\frac{\pi}{4}\right)$

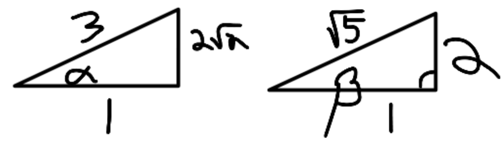
156. $\cos\left(\operatorname{arcsec}(3)+\arctan(2)\right)$



$$\arcsin\left(\frac{5}{13}\right)=\alpha$$
$$\sin(\alpha)=\frac{5}{13}$$

$$\sin(\alpha+\beta)=\sin\left(\alpha+\frac{\pi}{4}\right)=\sin\alpha\cos\beta+\cos\alpha\sin\beta=$$

$$\left(\frac{5}{13}\right)\left(\frac{1}{\sqrt{2}}\right)+\left(\frac{12}{13}\right)\left(\frac{1}{\sqrt{2}}\right)=\frac{17}{13\sqrt{2}}$$



$$\cos(\operatorname{arcsec}(3) + \arctan(2))$$

$$\sec(\alpha) = 3, \tan(\beta) = 2$$

$$\cos(\alpha+\beta) = \cos\alpha\cos\beta + \sin\alpha\sin\beta$$

$$\left(\frac{1}{3}\right)\left(\frac{1}{\sqrt{5}}\right) + \left(\frac{2\sqrt{2}}{3}\right)\left(\frac{2}{\sqrt{5}}\right) = \frac{1+4\sqrt{2}}{3\sqrt{5}}$$

Verify the identity:

$$\sin(\alpha+\beta) - \sin(\alpha-\beta) = 2\cos(\alpha)\sin(\beta)$$

$$(\sin\alpha\cos\beta + \cos\alpha\sin\beta) - (\sin\alpha\cos\beta - \cos\alpha\sin\beta) =$$
$$\sin\alpha\cos\beta + \cos\alpha\sin\beta - \sin\alpha\cos\beta + \cos\alpha\sin\beta =$$
$$2\cos(\alpha)\sin(\beta)$$

QED.

Verify the identity:

$$\frac{\sin(\alpha+\beta)}{\sin(\alpha-\beta)} = \frac{1+\cot(\alpha)\tan(\beta)}{1-\cot(\alpha)\tan(\beta)}$$

$$\frac{\sin\alpha\cos\beta + \cos\alpha\sin\beta}{\sin\alpha\cos\beta - \cos\alpha\sin\beta} = \frac{\sin\alpha\cos\beta + \cos\alpha\sin\beta}{\sin\alpha\cos\beta - \cos\alpha\sin\beta}\left(\frac{\frac{1}{\sin\alpha\cos\beta}}{\frac{1}{\sin\alpha\cos\beta}}\right)$$

$$= \frac{\sin\alpha\cos\beta\left(\frac{1}{\sin\alpha\cos\beta}\right) + \cos\alpha\sin\beta\left(\frac{1}{\sin\alpha\cos\beta}\right)}{\sin\alpha\cos\beta\left(\frac{1}{\sin\alpha\cos\beta}\right) - \cos\alpha\sin\beta\left(\frac{1}{\sin\alpha\cos\beta}\right)} = \frac{1+\left(\frac{\cos\alpha\sin\beta}{\sin\alpha\cos\beta}\right)}{1-\left(\frac{\cos\alpha\sin\beta}{\sin\alpha\cos\beta}\right)}$$

$$= \frac{1+\left(\frac{\cos\alpha}{\sin\alpha}\right)\left(\frac{\sin\beta}{\cos\beta}\right)}{1-\left(\frac{\cos\alpha}{\sin\alpha}\right)\left(\frac{\sin\beta}{\cos\beta}\right)} = \frac{1+\cot\alpha\tan\beta}{1-\cot\alpha\tan\beta}$$

QED.

Product to Sum formulas (obtained by rearranging the sum and difference formulas)

$$\cos\alpha\cos\beta = \frac{1}{2}[\cos(\alpha+\beta) + \cos(\alpha-\beta)]$$

$$\sin\alpha\sin\beta = -\frac{1}{2}[\cos(\alpha+\beta) - \cos(\alpha-\beta)]$$

$$\sin\alpha\cos\beta = \frac{1}{2}[\sin(\alpha+\beta) + \sin(\alpha-\beta)]$$

If your product is $\cos\alpha\sin\beta$, just switch the order and apply the bottom formula.

Apply the identity to rewrite the product as a sum.

Example.

$$\cos(3\theta)\cos(5\theta) = \frac{1}{2}[\cos(3\theta + 5\theta) + \cos(3\theta - 5\theta)] = \frac{1}{2}[\cos(8\theta) + \cos(-2\theta)] = \frac{1}{2}(\cos(8\theta) + \cos(2\theta))$$

Example.

$$\sin(3\theta)\sin(2\theta) = -\frac{1}{2}[\cos(\alpha + \beta) - \cos(\alpha - \beta)] = \frac{1}{2}[\cos(\alpha - \beta) - \cos(\alpha + \beta)] =$$

$$-\frac{1}{2}[\cos(5\theta) - \cos(\theta)] = \frac{1}{2}[\cos\theta - \cos(5\theta)]$$

These sum to product formulas also exist, but to derive them we would need additional identities we will work with in the next course. They are uncommon, and I won't be testing on them in this class. I include them here for completeness.

Sum to Product Formula

$$\cos x + \cos y = 2\cos\frac{x+y}{2}\cos\frac{x-y}{2}$$

$$\cos x - \cos y = -2\sin\frac{x+y}{2}\sin\frac{x-y}{2}$$

$$\sin x + \sin y = 2\sin\frac{x+y}{2}\cos\frac{x-y}{2}$$

$$\sin x - \sin y = 2\cos\frac{x+y}{2}\sin\frac{x-y}{2}$$

Next time: we'll do the double angle, half-angle, and power-reducing identities.