3/13/2025

Double-Angle Formulas, Half-Angle Identities, Power-Reducing Identities

Double Angle Formulas

$$\sin(2\theta) = 2\sin\theta\cos\theta$$
$$\cos 2\theta = \cos^2\theta - \sin^2\theta = 2\cos^2\theta - 1 = 1 - 2\sin^2\theta$$
$$\tan 2\theta = \frac{2\tan\theta}{1-\tan^2\theta}$$

Power-reducing Identities (derived from the cosine double angle formula, tangent by dividing the results)

$$\sin^2\theta = \frac{1}{2}(1-\cos 2\theta)$$

$$\cos^2\theta = \frac{1}{2}(1+\cos 2\theta)$$

$$\tan^2\theta = \frac{1-\cos 2\theta}{1+\cos 2\theta}$$

Half-Angle Formulas (derived from the power-reducing identities, by replacing the double angle $2\theta$ with $\alpha$, and thus, $\theta = \frac{\alpha}{2}$, then square rooting; the sign is determined by the quadrant the resulting angle is in).

$$\sin\left(\frac{\alpha}{2}\right) = \pm\sqrt{\frac{1-\cos\alpha}{2}}$$

$$\cos\left(\frac{\alpha}{2}\right) = \pm\sqrt{\frac{1+\cos\alpha}{2}}$$

$$\tan\left(\frac{\alpha}{2}\right) = \pm\sqrt{\frac{1-\cos\alpha}{1+\cos\alpha}}$$

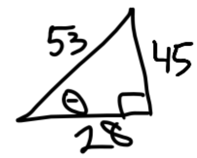
Example.

$$\cos(75°) = +\sqrt{\frac{1+\cos 150°}{2}} = \sqrt{\frac{1-\frac{\sqrt{3}}{2}}{2}\left(\frac{2}{2}\right)} = \sqrt{\frac{1-\sqrt{3}}{4}} = \frac{\sqrt{1-\sqrt{3}}}{2}$$
$$\frac{\alpha}{2} = 75° \rightarrow \alpha = 150°$$

Example.

$$\tan\left(\frac{7\pi}{8}\right) = -\sqrt{\frac{1-\cos\left(\frac{7\pi}{4}\right)}{1+\cos\left(\frac{7\pi}{4}\right)}} = \sqrt{\frac{1-\frac{1}{\sqrt{2}}}{1+\frac{1}{\sqrt{2}}}\left(\frac{\sqrt{2}}{\sqrt{2}}\right)} = \sqrt{\frac{\sqrt{2}-1}{\sqrt{2}+1}}$$

Example.
Given that $\cos(\theta) = \frac{28}{53}, in\ Q1.$
Find $\sin(2\theta)$, $\cos(2\theta)$, $\tan(2\theta)$




$$\sin(2\theta) = 2\sin(\theta)\cos(\theta) = 2\left(\frac{45}{53}\right)\left(\frac{28}{53}\right) = \frac{2520}{2809}$$

$$\cos 2\theta = \cos^2\theta - \sin^2\theta = \left(\frac{28}{53}\right)^2 - \left(\frac{45}{53}\right)^2 = \frac{784}{2809} - \frac{2025}{2809} = -\frac{1241}{2809}$$

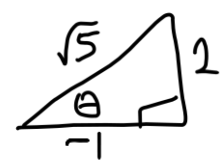
$$\cos 2\theta = 2\cos^2\theta - 1 = 2\left(\frac{28}{53}\right)^2 - 1 = \frac{1568}{2809} - \frac{2809}{2809} = -\frac{1241}{2809}$$

$$\cos 2\theta = 1 - 2\sin^2\theta = 1 - 2\left(\frac{45}{53}\right)^2 = 1 - \frac{4050}{2809} = \frac{2809}{2809} - \frac{4050}{2809} = -\frac{1241}{2809}$$

$$\tan(2\theta) = \frac{2\tan\theta}{1-\tan^2\theta} = \frac{2\left(\frac{45}{28}\right)}{1-\left(\frac{45}{28}\right)^2} = \frac{\frac{90}{28}}{1-\frac{2025}{784}} = \frac{\frac{90}{28}}{\frac{784}{784}-\frac{2025}{784}} = \frac{\frac{90}{28}}{-\frac{1241}{784}}\left(\frac{784}{784}\right) = \frac{2520}{-1241} =$$

$$-\frac{2520}{1241}$$

Example.
Given $\tan(\theta) = -2, in\ Q2$
Find $\sin(2\theta)$, $\sin\left(\frac{\theta}{2}\right)$




$$\sin(2\theta) = 2\sin\theta\cos\theta = 2\left(\frac{2}{\sqrt{5}}\right)\left(-\frac{1}{\sqrt{5}}\right) = -\frac{4}{5}$$

$$\sin\left(\frac{\theta}{2}\right) = +\sqrt{\frac{1-\cos\theta}{2}} = \sqrt{\frac{1-\left(-\frac{1}{\sqrt{5}}\right)}{2}\left(\frac{\sqrt{5}}{\sqrt{5}}\right)} = \sqrt{\frac{\sqrt{5}+1}{2\sqrt{5}}}$$

Verify the identity.

$$(\cos\theta - \sin\theta)^2 = 1 - \sin 2\theta$$

$$(\cos\theta - \sin\theta)(\cos\theta - \sin\theta) = \cos^2\theta - \cos\theta\sin\theta - \sin\theta\cos\theta + \sin^2\theta =$$

$$(\cos^2\theta + \sin^2\theta) - 2\sin\theta\cos\theta = 1 - \sin 2\theta$$

(supposed to be problem 60 in 10.4)

Example.

$$\csc(2\theta) = \frac{\cot\theta + \tan\theta}{2}$$

Left only →

$$\csc(2\theta) = \frac{1}{\sin(2\theta)} = \frac{1}{2\sin\theta\cos\theta} = \frac{1}{2}\left(\frac{1}{\sin\theta\cos\theta}\right)$$

Right only →

$$\frac{1}{2}(\cot\theta + \tan\theta) = \frac{1}{2}\left(\frac{\cos\theta}{\sin\theta} + \frac{\sin\theta}{\cos\theta}\right) = \frac{1}{2}\left(\frac{\cos\theta}{\sin\theta}\left(\frac{\cos\theta}{\cos\theta}\right) + \frac{\sin\theta}{\cos\theta}\left(\frac{\sin\theta}{\sin\theta}\right)\right) =$$

$$\frac{1}{2}\left(\frac{\cos^2\theta}{\sin\theta\cos\theta} + \frac{\sin^2\theta}{\sin\theta\cos\theta}\right) = \frac{\frac{1}{2}(\cos^2\theta + \sin^2\theta)}{\sin\theta\cos\theta} = \frac{1}{2}\left(\frac{1}{\sin\theta\cos\theta}\right)$$

Example.

$$32\sin^2\theta\cos^4\theta = 2 + \cos(2\theta) - 2\cos(4\theta) - \cos(6\theta)$$

$$32\sin^2\theta\cos^4\theta = 32\sin^2\theta\,(\cos^2\theta)(\cos^2\theta) =$$
$$32\left(\frac{1}{2}(1 - \cos 2\theta)\right)\left(\frac{1}{2}(1 + \cos 2\theta)\right)\left(\frac{1}{2}(1 + \cos 2\theta)\right)$$

$$\frac{32}{8}(1 - \cos 2\theta)(1 + \cos 2\theta)(1 + \cos 2\theta) = 4(1 - \cos^2 2\theta)(1 + \cos 2\theta) =$$

$$4(1 + \cos 2\theta - \cos^2 2\theta - \cos^3 2\theta) = 4 + 4\cos 2\theta - 4\cos^2 2\theta - 4\cos^3 2\theta =$$

$$4 + 4\cos 2\theta - 4\left(\frac{1}{2}(1 + \cos 4\theta)\right) - 4\cos^3 2\theta = 4 + 4\cos 2\theta - 2 - 2\cos 4\theta - 4\cos^3 2\theta =$$

$$2 + 4\cos 2\theta - 2\cos 4\theta - 4(\cos^2 2\theta)(\cos 2\theta)$$

$$= 2 + 4\cos 2\theta - 2\cos 4\theta - 4\left(\frac{1}{2}(1 + \cos 4\theta)\right)(\cos 2\theta) =$$

$$2 + 4\cos 2\theta - 2\cos 4\theta - (2 + 2\cos 4\theta)\cos(2\theta) =$$

$$2 + 4\cos 2\theta - 2\cos 4\theta - 2\cos 2\theta - 2\cos 4\theta\cos 2\theta =$$
$$2 + 2\cos 2\theta - 2\cos 4\theta - 2\cos 4\theta\cos 2\theta =$$

$$\cos(a)\cos(b) = \frac{1}{2}\left(\cos(a + b) + \cos(a - b)\right)$$

$$2 + 2\cos 2\theta - 2\cos 4\theta - 2\left[\frac{1}{2}(\cos(4\theta + 2\theta) + \cos(4\theta - 2\theta))\right] =$$

$$2 + 2\cos 2\theta - 2\cos 4\theta - \cos(6\theta) - \cos(2\theta) = 2 + \cos(2\theta) - 2\cos 4\theta - \cos(6\theta)$$

QED.

Example.

$$\frac{1}{\cos\theta - \sin\theta} + \frac{1}{\cos\theta + \sin\theta} = \frac{2\cos\theta}{\cos 2\theta}$$

Find a common denominator and apply identities.

Quiz due tonight.
Next week is spring break.
Resume on 3/25
When we come back, we'll solve trig equations.