4/10/2025

Complex Numbers in Polar Form
De Moivre's Theorem

Review of complex numbers

$$i = \sqrt{-1}$$
$$i^2 = -1$$
$$i^3 = -i$$
$$i^4 = 1$$
$$i^5 = i$$

Standard form for complex numbers

$$a + bi$$

$$\sqrt{-4} = \sqrt{-1}\sqrt{4} = 2i$$

$$\sqrt{-8}\sqrt{-6} = \sqrt{8}i\sqrt{6}i = \sqrt{48}(i^2) = -4\sqrt{3}$$

Complex conjugates

$$z = a + bi$$
$$\bar{z} = a - bi$$

Magnitude of a complex number

$$|z| = \sqrt{a^2 + b^2}$$

$$|z| = |\bar{z}|$$

Operations on complex numbers in standard form:

Addition:
Add component by component (add the real parts, add the imaginary parts)

$$3 + 2i + 5 - 6i = 8 - 4i$$

Subtraction:

$$(3 + 2i) - (5 - 6i) = 3 + 2i - 5 + 6i = -2 + 8i$$

Multiplication: multiply like binomial (FOIL)

$$(3 + 2i)(5 - 6i) = 15 - 18i + 10i - 12i^2 = 15 - 8i + 12 = 27 - 8i$$

Division:

$$\frac{3 + 2i}{5 - 6i}\left(\frac{5 + 6i}{5 + 6i}\right) = \frac{15 + 18i + 10i + 12i^2}{25 - 36i^2} = \frac{15 + 18i + 10i - 12}{25 + 36} = \frac{3 + 28i}{61} = \frac{3}{61} + \frac{28}{61}i$$

Real part of a complex number:

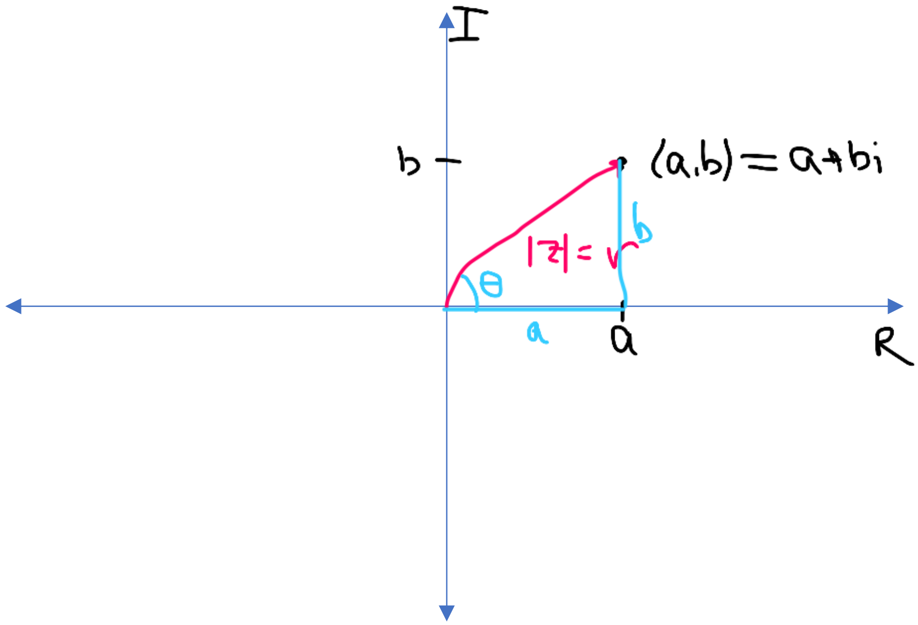
$$Re(z) = Re(a + bi) = a$$

Imaginary part of a complex number

$$Im(z) = Im(a + bi) = b$$

(imaginary part does not have i included when reporting)

The Complex Plane



With this idea in mind, we can apply our work with polar coordinates to complex numbers and find a representation of complex numbers in terms of their angle (with the real axis) and their magnitude.

$$z = r(\cos\theta + i\sin\theta)$$

$$a = r\cos\theta\,, b = r\sin\theta$$

Convert $z = 1 - i$ to polar form

$$a = 1, b = -1$$

All the formulas from last class for polar conversion still apply:

$$x^2 + y^2 = r^2$$
$$\theta = \tan^{-1}\left(\frac{y}{x}\right)$$
$$x = r\cos\theta$$
$$y = r\sin\theta$$

$$z = a + bi$$

$$a^2 + b^2 = r^2$$

$$\theta = \tan^{-1}\left(\frac{b}{a}\right)$$
$$a = r\cos\theta$$
$$b = r\sin\theta$$

$$r^2 = (1)^2 + (-1)^2 = 1 + 1 = 2$$
$$r = \sqrt{2}$$
$$\theta = \tan^{-1}\left(-\frac{1}{1}\right) = -\frac{\pi}{4}$$

$$z = \sqrt{2}\left(\cos\left(-\frac{\pi}{4}\right) + i\sin\left(-\frac{\pi}{4}\right)\right)$$

Exponential form:

$$z = re^{i\theta}$$

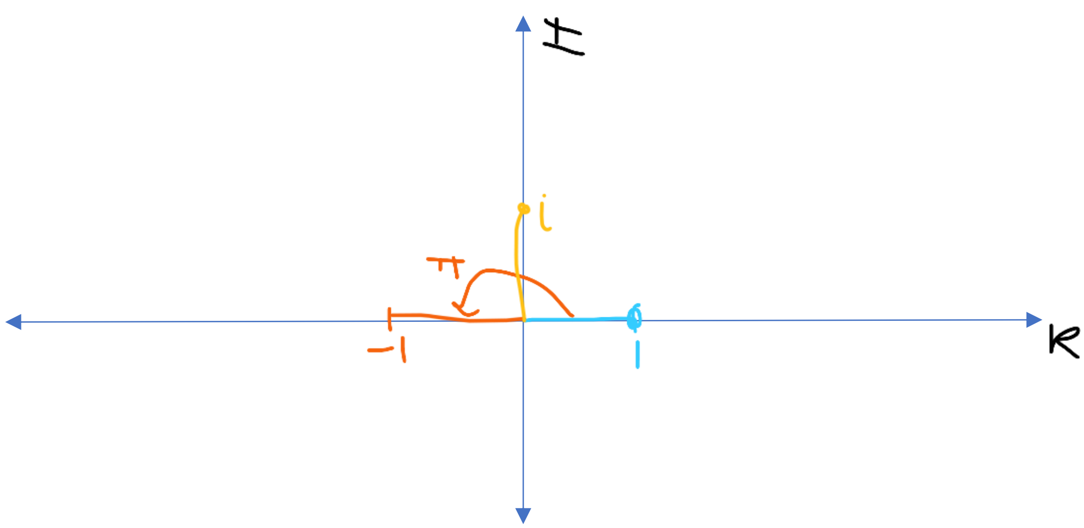
$$1 - i = \sqrt{2}e^{-\frac{\pi}{4}i}$$

Convert $2\left(\cos\left(\frac{\pi}{3}\right) + i\sin\left(\frac{\pi}{3}\right)\right)$ into rectangular form (standard form).

$$2\left(\frac{1}{2} + \frac{i\sqrt{3}}{2}\right) = 1 + \sqrt{3}i$$

In exponential form:

$$z = 2e^{\frac{\pi}{3}i}$$



$$e^{0i}$$

Length is 1, and the angle is 0

$$1(\cos 0 + i\sin 0) = 1 + 0i = 1$$

$$4e^{0i} = 4$$

$$e^{\frac{\pi}{2}i}$$

$$1\left(\cos\frac{\pi}{2} + i\sin\frac{\pi}{2}\right) = 0 + 1i = i$$

Pure imaginary number

$$e^{\frac{\pi}{2}i}\left(e^{\frac{\pi}{2}i}\right) = i(i) = i^2 = -1$$

$$e^{\frac{\pi}{2}i+\frac{\pi}{2}i} = e^{\pi i}$$

$$1(\cos\pi + i\sin\pi) = -1 + i(0) = -1$$

When I multiply in polar (exponential form) I add the angles together.
(if there is any magnitude, multiply those)

$$\left(3e^{\frac{\pi}{3}i}\right)\left(\frac{1}{2}e^{\frac{5\pi}{6}i}\right)$$

$$\left(3\times\frac{1}{2}\right)\left(e^{\frac{\pi}{3}i}\right)\left(e^{\frac{5\pi}{6}i}\right) = \frac{3}{2}e^{\left(\frac{\pi}{3}+\frac{5\pi}{6}\right)i} = \frac{3}{2}e^{\left(\frac{7\pi}{6}i\right)}$$

$$\frac{3}{2}\left(\cos\left(\frac{7\pi}{6}\right) + i\sin\left(\frac{7\pi}{6}\right)\right) = \frac{3}{2}\left(-\frac{\sqrt{3}}{2} - \frac{1}{2}i\right) = -\frac{3\sqrt{3}}{4} - \frac{3}{4}i$$

Check:

$$3e^{\frac{\pi}{3}i} = 3\left(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}\right) = 3\left(\frac{1}{2} + \frac{\sqrt{3}}{2}i\right) = \frac{3}{2} + \frac{3\sqrt{3}}{2}i$$

$$\frac{1}{2}e^{\frac{5\pi}{6}i} = \frac{1}{2}\left(\cos\left(\frac{5\pi}{6}\right) + i\sin\left(\frac{5\pi}{6}\right)\right) = \frac{1}{2}\left(-\frac{\sqrt{3}}{2} + \frac{1}{2}i\right) = -\frac{\sqrt{3}}{4} + \frac{1}{4}i$$

$$\left(\frac{3}{2} + \frac{3\sqrt{3}}{2}i\right)\left(-\frac{\sqrt{3}}{4} + \frac{1}{4}i\right) = \left(\frac{3}{2}\right)\left(-\frac{\sqrt{3}}{4}\right) + \left(\frac{3}{2}\right)\left(\frac{1}{4}i\right) - \left(\frac{3\sqrt{3}}{2}i\right)\left(\frac{\sqrt{3}}{4}\right) + \left(\frac{3\sqrt{3}}{2}i\right)\left(\frac{1}{4}i\right) =$$

$$-\frac{3\sqrt{3}}{8} + \frac{3}{8}i - \frac{9}{8}i - \frac{3\sqrt{3}}{8} = -\frac{3\sqrt{3}}{4} - \frac{3}{4}i$$

For multiplication: use the form provided (if the powers are small, or you are only multiplying two values).
For addition, you must go back to standard form.

Division in polar form:

$$\frac{\left(3e^{\frac{\pi}{3}i}\right)}{\frac{1}{2}e^{\frac{5\pi}{6}i}}$$

Division with powers (of the same base), subtract the exponents:

$$\frac{3}{\frac{1}{2}} \times \left(e^{\frac{\pi}{3}i - \frac{5\pi}{6}i}\right) = 6e^{-\frac{\pi}{2}i}$$

$$6\left(\cos\left(-\frac{\pi}{2}\right) + i\sin\left(-\frac{\pi}{2}\right)\right) = 6\big(0 + i(-1)\big) = -6i$$

Check:

$$\frac{\frac{3}{2} + \frac{3\sqrt{3}}{2}i}{-\frac{\sqrt{3}}{4} + \frac{1}{4}i} = \frac{\frac{3}{2} + \frac{3\sqrt{3}}{2}i}{-\frac{\sqrt{3}}{4} + \frac{1}{4}i}\left(\frac{4}{4}\right) = \frac{6 + 6\sqrt{3}i}{-\sqrt{3} + i}\left(\frac{-\sqrt{3} - i}{-\sqrt{3} - i}\right) = \frac{-6\sqrt{3} - 6i - 18i + 6\sqrt{3}}{3 + 1} = -\frac{24i}{4} = -6i$$

Powers of complex numbers

De Moivre's Theorem

$$z = a + bi = r\,e^{\theta i}$$

$$z^n = r^n e^{n\theta i}$$

$$z = 1 - i$$

Find $(1 - i)^3$

Recall from earlier: $1 - i = \sqrt{2}e^{-\frac{\pi}{4}i}$

$$\left(\sqrt{2}e^{-\frac{\pi}{4}i}\right)^3 = \left(\sqrt{2}\right)^3\left(e^{-\frac{3\pi}{4}i}\right) = 2\sqrt{2}\left(\cos\left(-\frac{3\pi}{4}\right) + i\sin\left(-\frac{3\pi}{4}\right)\right) = 2\sqrt{2}\left(-\frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}}i\right) = -2 - 2i$$

$$(1 - i)^3 = (1 - i)(1 - i)(1 - i) = (1 - i - i - 1)(1 - i) = (-2i)(1 - i) = (-2i + 2i^2) = -2 - 2i$$

Find $(1 - i)^7$

$$\left(\sqrt{2}e^{-\frac{\pi}{4}i}\right)^7 = \left(\sqrt{2}\right)^7 e^{\frac{-7\pi}{4}i} = \sqrt{128}\left(\cos\left(-\frac{7\pi}{4}\right) + i\sin\left(-\frac{7\pi}{4}\right)\right) = 8\sqrt{2}\left(\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}i\right) = 8 + 8i$$

Taking roots of complex numbers, for example $\sqrt{i}, \sqrt{1 - i}, \sqrt[3]{i}$, etc.

Where $z = r\,e^{\theta i}$

$$\sqrt[n]{z} = \sqrt[n]{r}\left(e^{\frac{\theta}{n}i}\right), \sqrt[n]{r}\left(e^{\frac{\theta+2\pi}{n}i}\right), \sqrt[n]{r}\left(e^{\frac{\theta+4\pi}{n}i}\right), etc.$$

Until you have the required number of roots.

Find $\sqrt{1-i}$

$$1 - i = \sqrt{2}e^{-\frac{\pi}{4}i}$$

We need two roots

$$\sqrt{1-i} = \left(\sqrt{2}\right)^{\frac{1}{2}} e^{-\frac{\pi}{4}\left(\frac{1}{2}\right)i} = \sqrt[4]{2}e^{-\frac{\pi}{8}i}$$

Second root:

$$\left(\sqrt{2}\right)^{\frac{1}{2}} e^{\left(-\frac{\pi}{4}+2\pi\right)\left(\frac{1}{2}\right)i} = \sqrt[4]{2}e^{\frac{7\pi}{8}i}$$

Find $\sqrt[3]{i}$, if $z = i = e^{\frac{\pi}{2}i}$

First root:

$$\left(e^{\frac{\pi}{2}i}\right)^{\frac{1}{3}} = e^{\frac{\pi}{6}i} = \cos\left(\frac{\pi}{6}\right) + i\sin\left(\frac{\pi}{6}\right) = \frac{\sqrt{3}}{2} + \frac{1}{2}i$$

Second root:

$$\left(e^{\left(\frac{\pi}{2}+2\pi\right)i}\right)^{\frac{1}{3}} = e^{\frac{5\pi}{6}i} = \cos\left(\frac{5\pi}{6}\right) + i\sin\left(\frac{5\pi}{6}\right) = -\frac{\sqrt{3}}{2} + \frac{1}{2}i$$

Third root:

$$\left(e^{\left(\frac{\pi}{2}+4\pi\right)i}\right)^{\frac{1}{3}} = e^{\frac{9\pi}{6}i} = e^{\frac{3\pi}{2}i} = \cos\left(\frac{3\pi}{2}\right) + i\sin\left(\frac{3\pi}{2}\right) = 0 - 1i = -i$$

$$\arg(z) = \theta$$

$$cis(\theta) = \cos\theta + i\sin\theta$$