4/15/2025

Conic Sections:
Circles, Ellipses, Parabolas

(Chapter 7 in other online book)

Conic sections are derived from slicing a cone, and the type of shape you get is going to depend on the relative angle that you slice the cone with.

Right circular cone.
Vertical axis through the center.
If you take a slice through the cone that is perpendicular to the central axis, you get a circle.

If we slice the cone (0-degree is horizontal (i.e. perpendicular to the vertical axis) at a small angle, smaller than the angle that makes up the side of the cone), then we get an ellipse.

The parabola is going to come from slicing the cone at an angle equal to the angle of the cone itself. It also only touches one half of the cone.

The hyperbola is going to come from slicing the cone at an angle that is steeper than the side of the cone, and it intersects with both the top and bottom of the cone (not as steep as the vertical axis).

Circles.
A circle is a curve where every point on the curve is equal-distant from the center.

$$d = \sqrt{(x - x_0)^2 + (y - y_0)^2}$$

$$r^2 = (x - x_0)^2 + (y - y_0)^2$$

Standard form

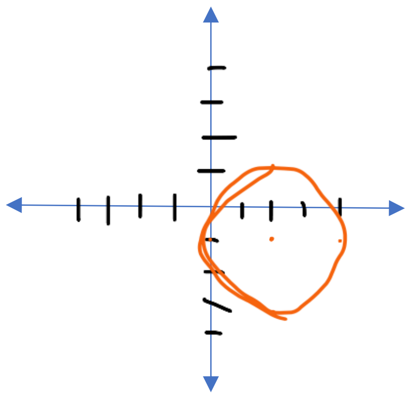
$$(x - h)^2 + (y - k)^2 = r^2$$

r is the radius and (h, k) is the center

Typically to plot the circle, we plot the center, and then select points r units from the center in the x-direction and r units in the y-direction, and then connect the dots.

$$(x - 2)^2 + (y + 1)^2 = 4$$

$$center\ (2, -1), radius = 2$$



Example.

$$x^2 - 4x + y^2 + 10y = -25$$

General form

$$Ax^2 + Bx^2 + Cx + Dy = E$$

$$(x^2 - 4x) + (y^2 + 10y) = -25$$

$$(x^2 - 4x + 4) + (y^2 + 10y + 25) = -25 + 4 + 25$$
$$x^2 + 2ax + a^2 = (x + a)^2$$

$$-4 = 2a$$
$$a = -2$$
$$a^2 = 4$$

$$2a = 10$$
$$a = 5$$
$$a^2 = 25$$

$$(x - 2)^2 + (y + 5)^2 = 4$$

$$center\ (2, -5), radius = 2$$

Example.

$$-2x^2 - 36x - 2y^2 - 112 = 0$$

$$x^2 + 18x + y^2 + 56 = 0$$

$$x^2 + 18x + y^2 = -56$$

$$(x^2 + 18x) + y^2 = -56$$
$$18 = 2a$$
$$a = 9$$
$$a^2 = 81$$

$$(x^2 + 18x + 81) + y^2 = -56 + 81$$
$$(x + 9)^2 + y^2 = 25$$

$$center\ (-9{,}0), radius = 5$$

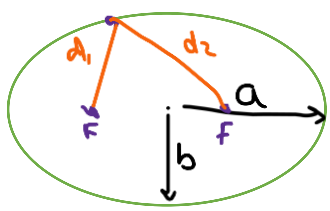
For general form to be a circle:

$$Ax^2 + By^2 + Cx + Dy + E = 0$$
$$A = B$$

Ellipse

Ellipse has a center, (h,k)
A major axis which is length 2a (distance is a in either direction from the center)
A minor axis which is length 2b (distance is b in either direction from the center)

On the major axis, there are two points called foci (focus is the singular)
Every point on the ellipse, the sum of the distances from two foci are constant.



$d_1 + d_2 = K$

$$\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} = 1$$

$$\frac{(x-h)^2}{b^2} + \frac{(y-k)^2}{a^2} = 1$$

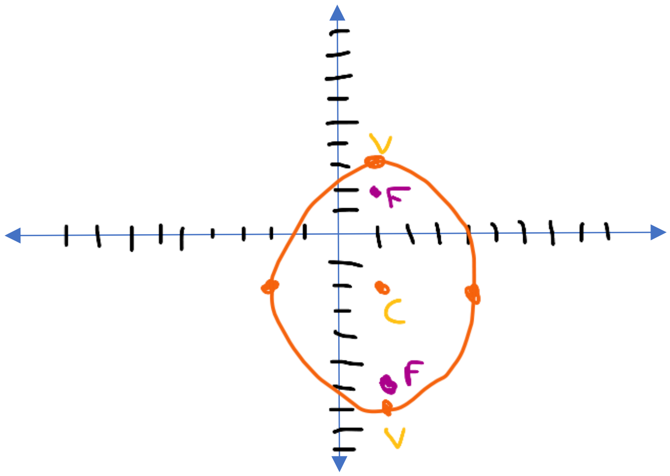
The only difference here is which direction is the longer one… a is always the larger value, and represents the semi-major axis.

The center is (h,k), and the vertex horizontal to the center is a units away (in the top equation, or b units away in the bottom equation), and the end of the minor axis is b unit away in the vertical direction (in the top equation or a units away in the bottom equation).

We identify the center. The coefficient under the x-term, square root it, and then plot that many units away horizontally from the center. Then the coefficient under the y-term, we square root it, and then go that many units up or down from the center. And then connect the dots.

Example.

$$\frac{(x-1)^2}{9} + \frac{(y+2)^2}{25} = 1$$



Foci:
There is a relationship between the distance to the focus and the major and minor axes.

$$a^2 = b^2 + c^2$$

c is the distance to the focus.

$$a^2 = 25$$
$$b^2 = 9$$

$$c^2 = a^2 - b^2$$
$$c^2 = 25 - 9 = 16$$
$$c = 4$$

The focus will be 4 units from the center on the major axis (in the same direction as a).

In some problems you will be given information on the focus and one other value (either a or b), and you can use that to find the one that is missing.

Example.
Find the standard equation of the ellipse given: foci $(\pm 3,0)$, length of the minor axis is 10

Foci are equal distance from the center, so the midpoint to obtain the center. (-3,0) and (3,0), so the midpoint is (0,0)

$$c = 3$$

Minor axis length = 2b

$$2b = 10$$
$$b = 5$$

$$a^2 = b^2 + c^2$$
$$a^2 = 5^2 + 3^2 = 25 + 9 = 41$$

$$a = \sqrt{41}$$

$$\frac{x^2}{41} + \frac{y^2}{25} = 1$$

Example.

$$5x^2 + 18y^2 - 30x + 72y + 27 = 0$$

Recall that in general form, $Ax^2 + By^2 + Cx + Dy + E = 0$, for an ellipse, A, B both be the same sign, but they don't have to be equal (shouldn't be)

$$(5x^2 - 30x) + (18y^2 + 72y) = -27$$
$$5(x^2 - 6x) + 18(y^2 + 4y) = -27$$

$$-6 = 2a$$
$$a = -3$$
$$a^2 = 9$$

$$4 = 2a$$
$$a = 2$$
$$a^2 = 4$$

$$5(x^2 - 6x + 9) + 18(y^2 + 4y + 4) = -27 + 5(9) + 18(4)$$

$$5(x-3)^2 + 18(y+2)^2 = 90$$

In the standard form of the ellipse, the constant is a 1, so divide by whatever constant value you have.

$$\frac{5(x-3)^2}{90} + \frac{18(y+2)^2}{90} = 1$$

$$\frac{(x-3)^2}{18} + \frac{(y+2)^2}{5} = 1$$

$$a^2 = 18, a = \sqrt{18} = 3\sqrt{2}$$
$$b^2 = 5, b = \sqrt{5}$$

Center is (3,-2)

$$c^2 = 18 - 5 = 13, c = \sqrt{13}$$

Eccentricity

$$e = \frac{c}{a}$$

c is the distance to the focus, a is the distance to the vertex (semi-major axis)

$$e = \frac{\sqrt{13}}{\sqrt{18}} \approx 0.8498 \ldots$$

For an ellipse, the eccentricity is always between 0 and 1 i.e. the open interval (0,1).

The eccentricity of a circle is given as 1 (we say the focus and the center are the same, therefore the distance to the focus is 0).

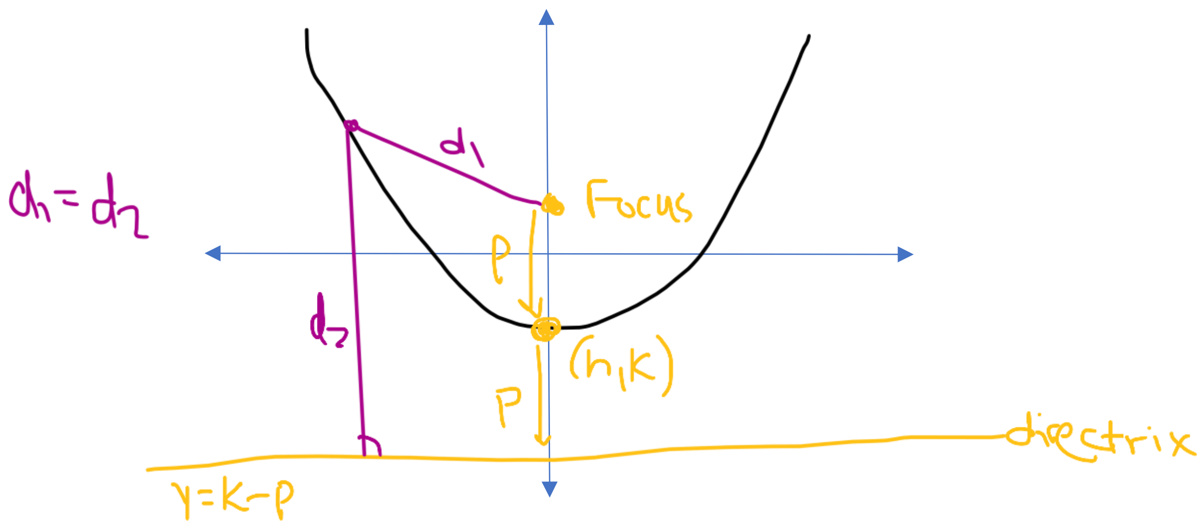
Parabola
(defined to have an eccentricity of 1 (vertex and the focus becoming the same length))

These are the only conics that can be functions

$$(x-h)^2 = 4p(y-k)$$

Linear y, and squared x (this is a function because we can solve for y)
Center (h,k) = vertex

p is the distance between what we call the vertex in a parabola (center) and what we call the focus



The parabola can be defined as curve which is equal-distant from the focus (d1) and the directrix (d2).

$$d_1 = d_2$$

The sideways opening parabola can be expressed in similar form:

$$4p(x-h) = (y-k)^2$$

y is squared, and so it's not a function.

If the coefficient in front of the linear term is positive, then the function version opens upward and the sideways version opens rightward. If the coefficient is negative, then the function version opens downward, and the sideways version opens to the left.

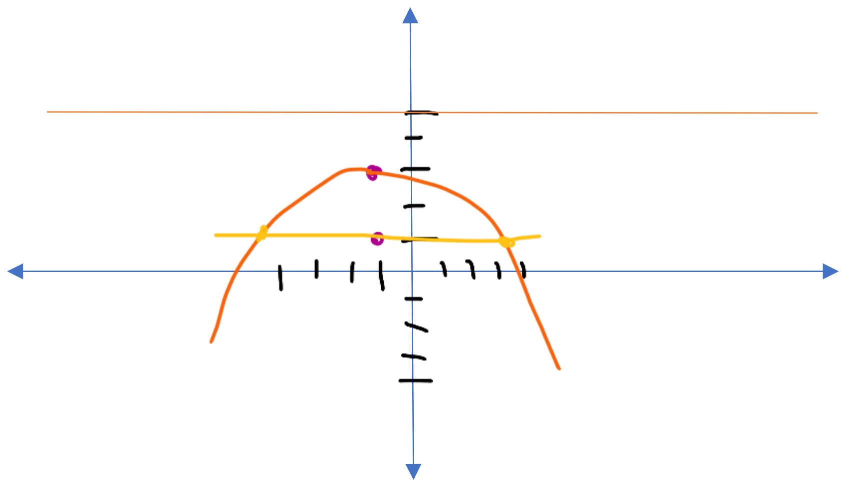
Example.

Graph $(x+1)^2 = -8(y-3)$

$$(h,k) = vertex = (-1,3)$$
$$4p = -8$$
$$p = -2$$

x^2 means opens up or down, and p is negative, so it open down.
The focus is inside the curve of the parabola
Go down two points from the vertex to get to the focus (-1,1) = (-1, 3-2)

Directrix will be a line p units in the opposite direction (here, up two units from the vertex) y=constant
Here, y=3+2=5

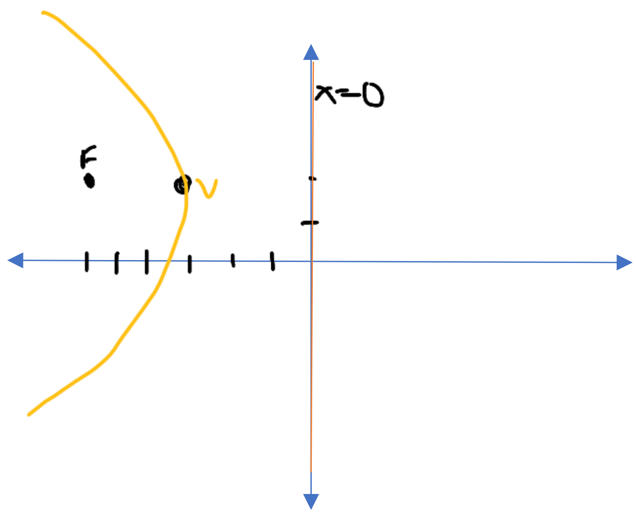


The latus rectum is a line that passes through the focus, formed by two points with the same y-value as the focus. Plug in the y-value for the focus and solve for the two x-values that go with it (in the upward/downward orientation).

Example.

$$(y-2)^2 = -12(x+3)$$

$$(h,k) = vertex = (-3,2)$$
$$4p = -12$$
$$p = -3$$



In general form, there is only one squared term.

$$y^2 - 10y - 27x + 133 = 0$$

Only the y-squared term. Opens sideways.
Have to complete the square
Put everything else on the other side of the equation

$$(y^2 - 10y) = 27x - 133$$
$$y^2 - 10y + 25 = 27x - 133 + 25$$
$$(y-5)^2 = 27x - 108$$

Factor the coefficient of the linear term on the linear side

$$(y-5)^2 = 27(x-4)$$

Vertex (4,5)

$$p = \frac{27}{4} = 6.75$$

Hyperbolas and conics in polar form on Thursday.