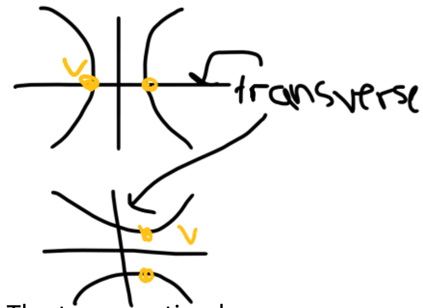
4/17/2025

Hyperbolas
Conic sections in polar form

Hyperbolas



$$\frac{(x-h)^2}{a^2} - \frac{(y-k)^2}{b^2} = 1$$

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The transverse axis is determined by the positive component of the equation. The top equation has a horizontal (or x-axis transverse axis), and the bottom one has a vertical (or y-axis transverse).

Center (h,k)
c is the distance to the focus
but the relationship between a, b, and c is different from the ellipse.
In the hyperbola, the distance to the focus is furthest: $c^2 = a^2 + b^2$

There is no rule about the relative sizes of a and b.

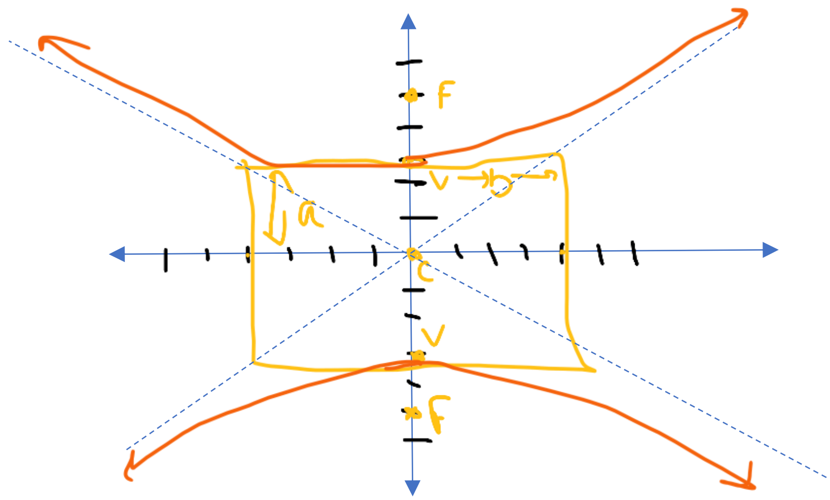
Vertex is the closest point on the hyperbola to the center. The point on the hyperbola where the two sections of the graph are the closest.

The center will be right in between them. Since we are using a as the scalar under the positive term, the distance from the vertex to the center will be a.

The asymptotes (linear, slanted lines going through the center) are given by:
The asymptotes are going to pass through the corners of a "box" defined by the lengths of a and b. The slopes will be $\pm\frac{a}{b}$ or $\pm\frac{b}{a}$ depending on the orientation of the hyperbola.

Example.

$$\frac{y^2}{9} - \frac{x^2}{16} = 1$$



Center (0,0), vertices (0,3),(0,-3)

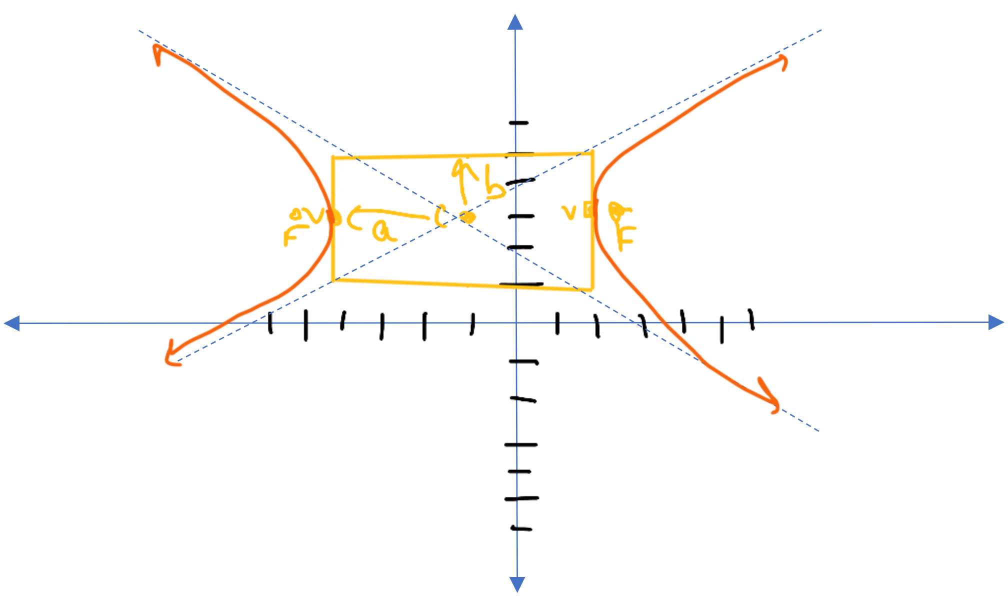
The asymptotes are $y = \pm\frac{3}{4}x$ $\left(y - k = \pm\frac{a}{b}(x + h)\right)$

$$c^2 = 9 + 16 = 25$$
$$c = 5$$

Focus points (0,5), (0,-5)

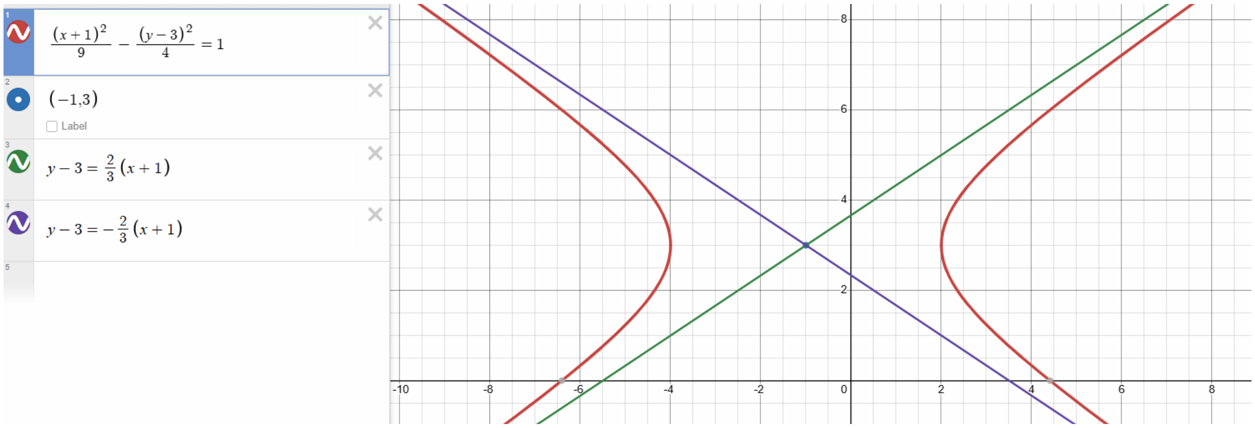
Example.

$$\frac{(x+1)^2}{9} - \frac{(y-3)^2}{4} = 1$$



$$c^2 = 9 + 4 = 13$$
$$c = \sqrt{13}$$

Asymptotes $y - 3 = \pm\frac{2}{3}(x + 1)$



General form:

$$Ax^2 + By^2 + Cx + Dy + E = 0$$

For a hyperbola, the A and B terms will have different signs. (They can have the same magnitude.) Complete the square similarly to how we did with the ellipse.

Eccentricity of a hyperbola is always $e > 1$, since $e = \frac{c}{a}$,and c is always longer in a hyperbola than is a.

Conic Sections in polar form.

Circles in polar form.

Simplest is centered at a point, with a fixed radius.
If centered at the origin:

$$r = a$$

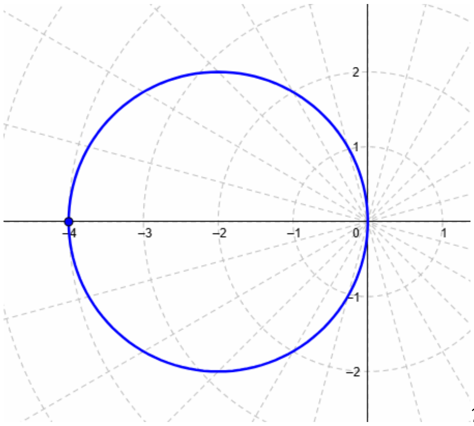
If you move this off the origin, use standard form and replace $x = r\cos\theta$, $y = r\sin\theta$
This will not necessarily be a function in polar coordinates. (you may not be able to solve for r)

If we move the center off the origin by the same size as the radius… (and only in x or y) then we'll get a nicer polar function

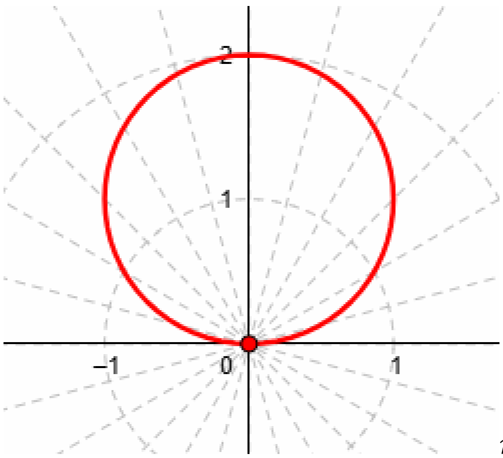
$$r = \pm 2a\sin\theta$$
$$r = \pm 2a\cos\theta$$

a is the radius.

https://www.geogebra.org/m/ApcfSCZY



$r = -4\cos\theta$



$r = 2\sin\theta$

Special cases, $e = 0$ for all circles.

Other conics

$$r = \frac{k}{1 \pm e\cos\theta} \; or \; r = \frac{k}{1 \pm e\sin\theta}$$

All ellipses, parabolas and hyperbolas all look like this.

What distinguishes one from the other is e (the coefficient of the sine or cosine)
Sine and cosine determine the orientation.

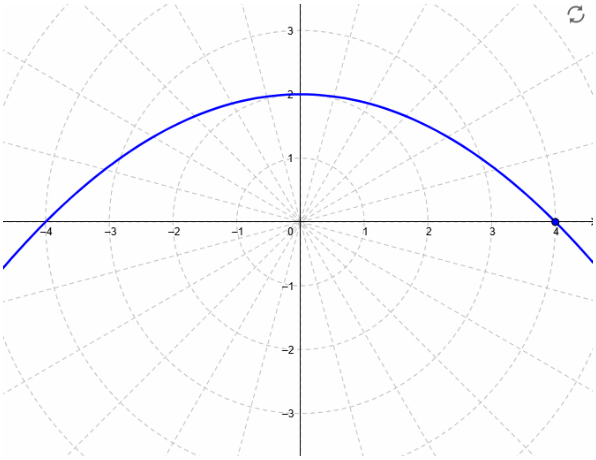
You may have to do some algebra to get that constant in the denominator to be 1.

Example.

$$r = \frac{4}{1 - \sin\theta}$$

$$e = 1$$

Therefore, this is a parabola.

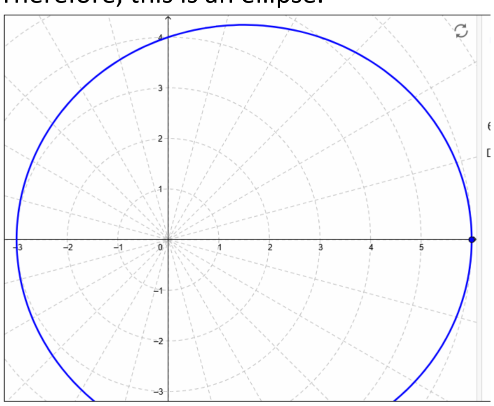


$$r = \frac{12}{3 - \cos\theta}$$

$$\frac{12\left(\frac{1}{3}\right)}{\frac{1}{3}(3 - \cos\theta)} = \frac{4}{1 - \frac{1}{3}\cos\theta}$$

$$e = \frac{1}{3}$$

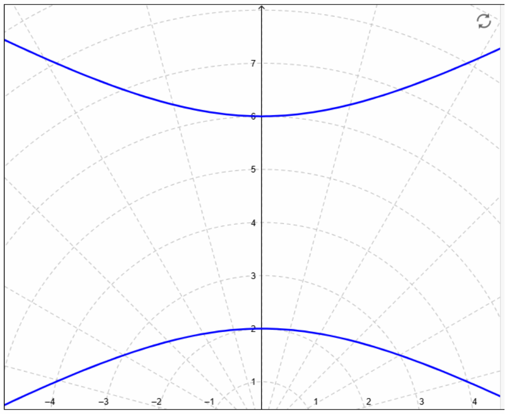
Therefore, this is an ellipse.



$$r = \frac{6}{1 + 2\sin\theta}$$

$$e = 2$$

Therefore this is a hyperbola.



Next section is Vectors?