4/22/2025

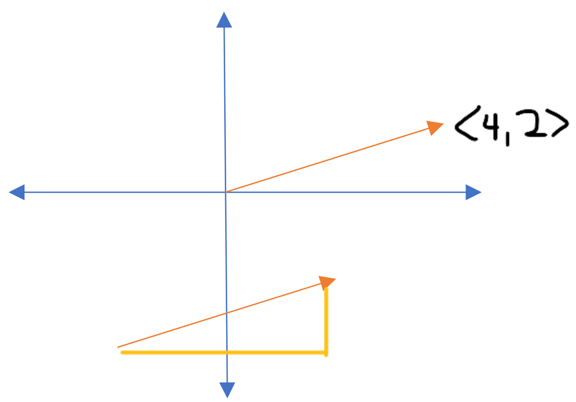
Vectors (11.8, 11.9)

Vectors are objects that have both magnitude and direction.
We can express these as magnitude and direction (radius is magnitude and theta is direction).
We can also express these in “component form”, where we describe how far we are going in each cardinal direction <x,y>.

$$\begin{bmatrix} x \\ y \end{bmatrix}, \langle x, y \rangle, \begin{pmatrix} x \\ y \end{pmatrix}, etc.$$

ijk notation

$$\langle x, y \rangle = x\hat{\imath} + y\hat{\jmath}$$



The vector <4,2> starts at the origin and ends at the point (4,2) in the plane. We are moving 4 units in the x-direction and 2 units in the y-direction.

If we start at the point (-2,-5), then the end point will be at (2,-3).

Find the vector that connect the points P(3,6) and Q(1,-2)

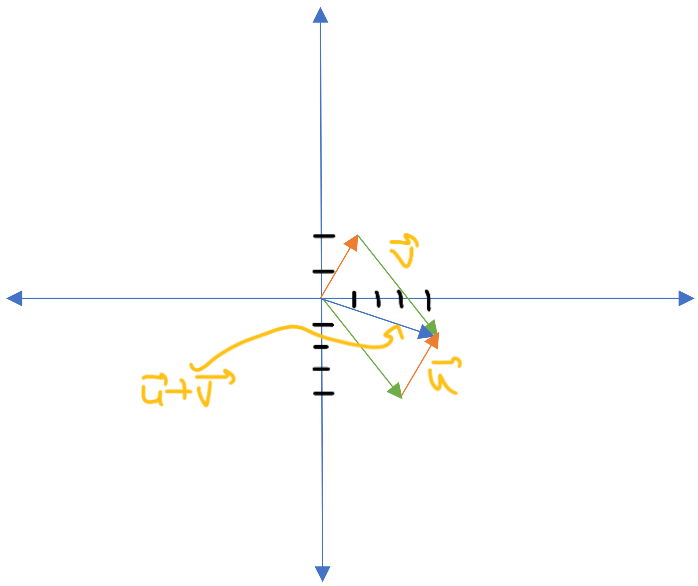
$$\overrightarrow{PQ} = \langle 1-3, -2-6 \rangle = \langle -2, -8 \rangle$$

$$\langle x, y \rangle = \langle r\cos\theta, r\sin\theta \rangle$$
$$r = \|\langle x, y \rangle\| = \sqrt{x^2 + y^2}$$
$$\theta = \tan^{-1}\left(\frac{y}{x}\right)$$

Adding vectors
Add vectors in component form: add them component by component

$$\vec{u} = \langle 1,2 \rangle, \vec{v} = \langle 3, -4 \rangle$$

Add the vectors: $\vec{u} + \vec{v} = \langle 1,2 \rangle + \langle 3, -4 \rangle = \langle 1+3, 2+(-4) \rangle = \langle 4, -2 \rangle$



Adding vectors by the parallelogram rule. The first vector is drawn from the origin. The second vector is drawn from the origin and from the endpoint of the first vector (that gives you parallel sides), and then the first is drawn from the endpoint of the second vector (second set of parallel sides). This gives us a parallelogram. And the sum of the vectors is the vector from the origin to the farthest point on the parallelogram.

Triangle rule:
You construct it like the top half of the parallelogram: the first vector starts at the origin. The second vector starts at the endpoint of the first vector. Then the sum is the vector from the origin to where the second vector stops.

Note: if you connect the two endpoints together, that is subtraction, not addition.

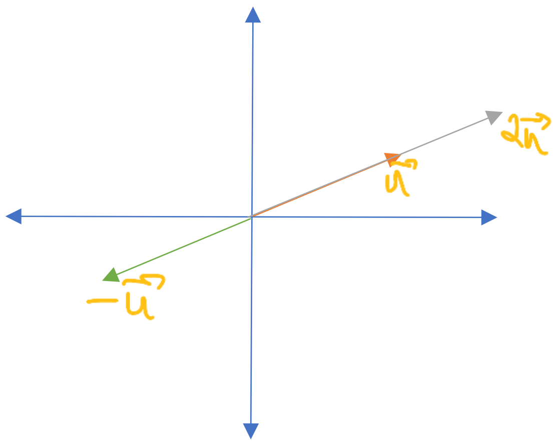
We can scale vectors: keeps the direction the same, but changes the magnitude.

$$k\vec{u} \to 2\vec{u} = 2\langle 1,2\rangle = \langle 2(1), 2(2)\rangle = \langle 2,4\rangle$$

Subtraction is defined as $\vec{u} - \vec{v} = \vec{u} + (-1)\vec{v}$

Example.

$$2\vec{u} - 3\vec{v} = 2\langle 1,2\rangle - 3\langle 3,-4\rangle = \langle 2,4\rangle - \langle 9,-12\rangle = \langle 2-9, 4-(-12)\rangle = \langle -7,16\rangle$$



Vector properties:

$$\vec{u} + \vec{v} = \vec{v} + \vec{u}$$
$$\vec{u} + (\vec{v} + \vec{w}) = (\vec{u} + \vec{v}) + \vec{w}$$
$$\vec{u} + \vec{0} = \vec{u} + \langle 0,0 \rangle = \vec{u}$$
$$\vec{u} + (-\vec{u}) = \vec{0}$$

$$k(r\vec{u}) = (kr)(\vec{u}) = r(k\vec{u})$$
$$1\vec{u} = \vec{u}$$
$$(k + r)\vec{u} = k\vec{u} + r\vec{u}$$
$$k(\vec{u} + \vec{v}) = k\vec{u} + k\vec{v}$$

$if\ k\vec{u} = 0$, then either $k = 0, or\ \vec{u} = \vec{0}$

$$\|\vec{u}\| = |\vec{u}| = \sqrt{x^2 + y^2}$$

Given the vector $\vec{u} = \langle -4,4 \rangle$
Find the magnitude and direction of the vector.

$$\|\vec{u}\| = \sqrt{(-4)^2 + (4)^2} = \sqrt{16 + 16} = \sqrt{32} = 4\sqrt{2}$$

Direction

$$\theta = \tan^{-1}\left(\frac{4}{-4}\right) = \tan^{-1}(-1) = -\frac{\pi}{4} + \pi = \frac{3\pi}{4}\ (135°)$$

Convert the vector with a magnitude of 50 units and a direction of $-30°$ relative to the positive x-axis to component form.

$$\langle r\cos\theta\,, r\sin\theta \rangle = \langle 50\cos(-30°)\,, 50\sin(-30°) \rangle = \langle 50\left(\frac{\sqrt{3}}{2}\right), 50\left(\frac{1}{2}\right) \rangle = \langle 25\sqrt{3}, 25 \rangle$$

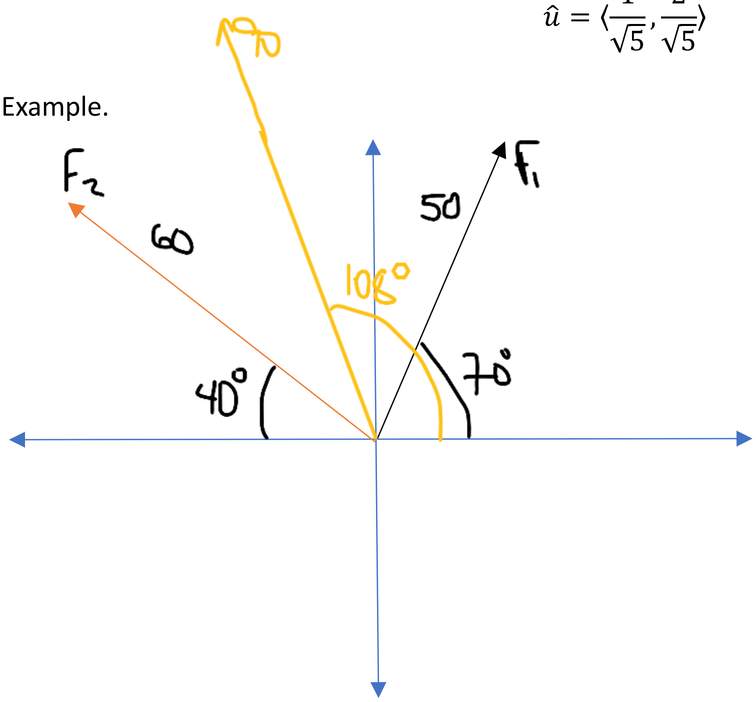
Sometimes we want a unit vector (a vector of length =1), but still pointed in the given direction.

$$\vec{u} = \langle 1,2 \rangle$$

The unit vector is given by $\hat{u} = \frac{\vec{u}}{\|\vec{u}\|}$

$$\|\vec{u}\| = \sqrt{1^2 + 2^2} = \sqrt{5}$$
$$\hat{u} = \langle \frac{1}{\sqrt{5}}, \frac{2}{\sqrt{5}} \rangle$$

Example.



Force #1 has a magnitude of 50 pounds and makes an angle with the positive x-axis of 70-degrees, and Force #2 has a magnitude of 60 pounds and makes an angle with the negative x-axis of 40-degrees. What is the resulting total force?
(resulting force is $F_1 + F_2 = F_{total}$)

$$\overrightarrow{F_1} = \langle 50\cos(70°), 50\sin(70°) \rangle = \langle 17.10100717, 46.98463104 \rangle$$
$$\overrightarrow{F_2} = \langle 60\cos(140°), 60\sin(140°) \rangle = \langle -45.96266659, 38.56725658 \rangle$$
$$\overrightarrow{F_{total}} = \langle 17.10100717 - -45.96266659, 46.98463104 + 38.56725658 \rangle = \langle -28.86165942, 85.55188762 \rangle$$

$$\left\|\overrightarrow{F_{total}}\right\| = \sqrt{(-28.86165942)^2 + (85.55188762)^2} = \sqrt{8152.12086} \approx 90.28909 \ldots$$

$$\theta = \tan^{-1}\left(\frac{85.55188762}{-28.86165942}\right) \approx -71.3577\ldots° + 180° = 108.64\ldots°$$

Dot Product
Sometimes called the Scalar product or inner product.

$$\vec{u} \cdot \vec{v}$$
$$\vec{u} = \langle u_1, u_2 \rangle, \vec{v} = \langle v_1, v_2 \rangle$$
$$\vec{u} \cdot \vec{v} = u_1 v_1 + u_2 v_2$$

Multiplying the components together (corresponding ones) and then adding the products.

$$\vec{u} = \langle 1,2 \rangle, \vec{v} = \langle 3,-4 \rangle$$
$$\vec{u} \cdot \vec{v} = 1(3) + 2(-4) = 3 - 8 = -5$$

The value is related to the angle between the vectors.

$$\cos(\theta_{between}) = \frac{\vec{u} \cdot \vec{v}}{\|\vec{u}\|\|\vec{v}\|}$$

$$\cos^{-1}\left(-\frac{5}{\sqrt{5}\sqrt{25}}\right) = \cos^{-1}\left(-\frac{5}{5\sqrt{5}}\right) = \cos^{-1}\left(-\frac{1}{\sqrt{5}}\right) = 116.565\ldots°, 2.03444\ldots radians$$

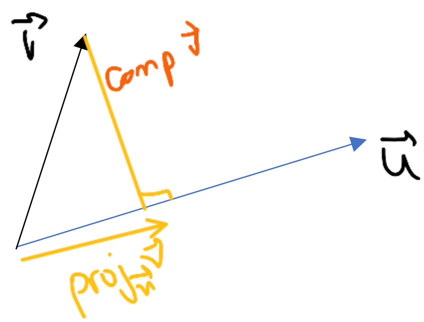
The angle is obtuse if the dot product is negative
The angle is acute if the dot product is positive.
The angle is a right angle is the dot product is zero: the two vectors are perpendicular

Dots products are commutative: $\vec{u} \cdot \vec{v} = \vec{v} \cdot \vec{u}$
Associative:$(k\vec{u}) \cdot \vec{v} = k(\vec{u} \cdot \vec{v}) = \vec{u} \cdot (k\vec{v})$
Distributive: $\vec{u} \cdot (\vec{v} + \vec{w}) = \vec{u} \cdot \vec{v} + \vec{u} \cdot \vec{w}$
Magnitude: $\|\vec{u}\|^2 = \vec{u} \cdot \vec{u}$

$$\|\vec{u}\| = \sqrt{\vec{u} \cdot \vec{u}}$$

Projection of one vector onto a second vector: here we are projecting $\vec{v}$ onto the vector $\vec{u}$

$$proj_{\vec{u}}\vec{v} = \left(\frac{\vec{u} \cdot \vec{v}}{\|\vec{u}\|^2}\right)\vec{u}$$



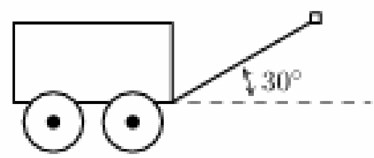
$$proj_{\vec{u}}\vec{v} = \left(\frac{\vec{u} \cdot \vec{v}}{\|\vec{u}\|^2}\right)\vec{u} = \left(-\frac{5}{5}\right)\langle 1,2 \rangle = (-1)\langle 1,2 \rangle = \langle -1,-2 \rangle$$

$$comp_{\vec{u}}\vec{v} = \vec{v} - proj_{\vec{u}}\vec{v} = \langle 3,-4 \rangle - \langle -1,-2 \rangle = \langle 4,-2 \rangle$$

Work.
If the force and direction of motion are pointing in the same direction, then $W = Fd$,

But if they are not pointing in the same direction, then $W = \vec{F} \cdot \vec{d}$

Example 11.9.5. Taylor exerts a force of 10 pounds to pull her wagon a distance of 50 feet over level ground. If the handle of the wagon makes a $30°$ angle with the horizontal, how much work did Taylor do pulling the wagon? Assume Taylor exerts the force of 10 pounds at a $30°$ angle for the duration of the 50 feet.



$$\vec{F} = \langle 10\cos 30°, 10\sin 30°\rangle = \langle 5\sqrt{3}, 5\rangle$$
$$\vec{d} = \langle 50,0\rangle$$

$W = \langle 5\sqrt{3}, 5\rangle \cdot \langle 50,0\rangle = 250\sqrt{3}$ foot-pounds

Next time: Parametric equations, and review for the exam (next Tuesday is the test).