5/1/2025

Sequences and Summation Notation
Mathematical Induction
(ch 9 in the precalc book)

Sequences are ordered lists of numerical values whose input variable is the set of natural numbers $\mathbb{N}$.
The natural numbers start at 1, 2, 3, 4,…
Some notations (depending on the book) will start at 0 (whole numbers).

Notation for sequences is a_n: $\{a_1, a_2, a_3, a_4 \dots\}$

$$a_n = f(n)$$

Explicit form used a function in terms of n to express the values in the sequence in a compact way.
But you can also use a recursive form:

$$a_{n+1} = f(a_n)$$

Or

$$a_{n+1} = f(a_n, a_{n-1}, \dots)$$

Generate 5 terms of the sequence from an explicit formula representing the sequence.

$$a_n = \frac{5^{n-1}}{3^n}, n \geq 1$$

$$a_n = \left\{\frac{1}{3}, \frac{5}{9}, \frac{25}{27}, \frac{125}{81}, \frac{625}{243}, \dots\right\}$$

One advantage of the explicit form is that we can go to any term in the sequence without generating all the prior terms in the sequence.

$$a_{10} = \frac{1953125}{59049}$$

Example.
Generating 5 terms of the sequence from the a_n sequence:

$$a_n = \left\{\frac{1 + (-1)^i}{i}\right\}_{i=2}^{\infty}$$

$$\left\{1, 0, \frac{1}{2}, 0, \frac{1}{3}, \dots\right\}$$

Recursive version: you need both the relationship between terms in the sequence, and you need a seed value (the initial value in the sequence.

Generate the first 5 terms of the sequence given by:

$$f_n = n \cdot f_{n-1}, f_0 = 1$$

Alternatively, this same sequence formula could be given by $f_{n+1} = (n+1) \cdot f_n$

$$\{1, 1, 2, 6, 24, \ldots\}$$

$$f_n = n!, n \geq 0$$
$$n! = n(n-1)(n-2) \ldots .(3)(2)(1)$$

factorials

$$7! = 7(6)(5)(4)(3)(2)(1)$$
$$0! = 1$$

Fibonacci sequence:

$$F_{n+1} = F_n + F_{n-1}, F_1 = 1, F_2 = 1$$

$$\{1, 1, 2, 3, 5, 8, 13, \ldots .\}$$

In recursive formulas, if I want to just to F_{10}, I need to generate all the values that precede F_{10}.

Arithmetic sequences, and geometric sequences.

Arithmetic sequences have a “common difference” between terms in the sequence
Geometric sequence has a “common ratio” between terms in the sequence

Examples:

$$a_n = \{3,7,11,15,19,23, \ldots\}$$

This is an arithmetic sequence because each term is +4 from the previous term.

Recursive formula: $a_n = a_{n-1} + 4, a_1 = 3$
Explicit formula: $a_n = (n-1)d + a_1 \ or \ a_n = nd + a_0$

$$a_n = 4(n-1) + 3 = 4n - 1$$

Geometric sequence

$$a_n = \{2, 6, 18, 54, 162, \ldots\}$$

Recursive: $a_n = a_{n-1}r$

$$a_n = 3a_{n-1}, a_0 = 2$$

Explicit: $a_n = a_0(r)^n$

$$a_n = 2(3)^n$$

Arithmetic sequences are based on linear functions in n, and geometric sequences are based on exponential functions in n.

Example:
Generate an explicit function for the sequence given by:

$$\left\{\frac{1}{2}, -\frac{3}{4}, \frac{9}{8}, -\frac{27}{16}, \ldots\right\}$$

When you have a fraction, it may help to find a pattern in the numerator and the denominator separately, and then combine them.

Numerator sequence: $\{1, -3, 9, -27, \dots\} = (-3)^n$
Denominator sequence: $\{2, 4, 8, 16, \dots\} = 2^n$

$$a_n = \frac{(-3)^n}{2^{n+1}}, n \geq 0$$

$$a_n = \frac{(-3)^{n-1}}{2^n}, n \geq 1$$

$$\left\{\frac{1}{1}, -\frac{3}{4}, \frac{9}{9}, -\frac{27}{16}, \frac{81}{25}, -\frac{243}{36}, \dots\right\}$$

Numerator: $(-3)^n$
Denominator: n^2

$$a_n = \frac{(-3)^n}{(n+1)^2}, n \geq 0$$

$$a_n = \frac{(-3)^{n-1}}{n^2}, n \geq 1$$

$$\left\{\frac{1}{1}, \frac{3}{2}, \frac{7}{6}, \frac{15}{24}, \frac{31}{120}, \dots\right\}$$

Numerator: $\{1,3,7,15,31, \dots\} = 2^n - 1$
Denominator: $\{1, 2, 6, 24, 120 \dots\} = n!$

$$\{1,3,7,15,31, \dots\} + 1$$
$$\{2,4,8,16,32, \dots\} = 2^n$$

$$a_n = \frac{2^n - 1}{n!}, n \geq 1$$

Common sequence components:

$$n^2 = 1, 4, 9, 16, 25, 36, \dots.$$
$$n^3 = 1, 8, 27, 64, 125, \dots$$
$$(-1)^n = 1, -1, 1, -1, 1 \dots$$
$$n^n = 1, 4, 27, 256, 3125, \dots$$
$$2^n = 1, 2, 4, 8, 16, 32, \dots$$
$$3^n = 1, 3, 9, 27, 81, \dots$$
$$n! = 1, 1, 2, 6, 24, 120, \dots$$
$$n = 1, 2, 3, 4, 5 \dots$$
$$2n + 1 = 1, 3, 5, 7, 9, 11, \dots$$
$$2n = 2, 4, 6, 8, 10, \dots$$

Summation notation

$$\sum_{i=0}^{n} a_i = a_0 + a_1 + a_2 + \cdots + a_n$$

Example:

$$\sum_{n=1}^{6} (2n-1) = 1 + 3 + 5 + 7 + 9 + 11 = 36$$

$$\sum_{i=1}^{6} (2i-1) = \sum_{j=1}^{6} (2j-1) = \sum_{k=1}^{6} (2k-1) = \sum_{n=1}^{6} (2n-1)$$

Find the sum:

$$\sum_{k=1}^{4} \frac{13}{100^k} = \frac{13}{100} + \frac{13}{10{,}000} + \frac{13}{1{,}000{,}000} + \frac{13}{100{,}000{,}000} = 0.13 + 0.0013 + 0.000013 + 0.00000013$$
$$= 0.13131313$$

Properties:

$$\sum_{n=0}^{N} (a_n \pm b_n) = \sum_{n=0}^{N} a_n \pm \sum_{n=0}^{N} b_n$$

$$\sum_{n=0}^{N} c a_n = c \sum_{n=0}^{N} a_n$$

$$\sum_{n=0}^{N} a_n = \sum_{n=0}^{p} a_n + \sum_{n=p+1}^{N} a_n \,, 0 \le p \le N$$

$$\sum_{n=m}^{p} a_n = \sum_{n=m+r}^{p+r} a_{n-r}$$

Finite sum of an arithmetic sequence:

$$S = n\left(\frac{(a_1 + a_n)}{2}\right) = \frac{n}{2}(a_1 + a_n)$$

Geometric sequences:

$$S = \frac{(a_1 + a_{n+1})}{1-r}$$

If $|r| < 1$, then

$$\sum_{n=0}^{\infty} ar^n = \frac{a}{1-r}$$

(Zeno's paradox)

Write $0.\overline{26}$ as a summation

$$0.\overline{26} = 0.262626262626\,...$$

$$0.26 + 0.0026 + 0.000026 + 0.00000026 + \cdots$$

$$\frac{26}{100} + \frac{26}{10,000} + \frac{26}{1,000,000} + \frac{26}{100,000,000} + \cdots$$

$$\frac{26}{100^1} + \frac{26}{100^2} + \frac{26}{100^3} + \frac{26}{100^4} + \cdots$$

$$\sum_{n=1}^{\infty} \frac{26}{100^n} = 26 \sum_{n=1}^{\infty} \left(\frac{1}{100}\right)^n$$

$$\frac{26}{99}$$

Mathematical induction is a proof technique for working with sequences.
Step 1) prove that the formula works for some initial values (n=1)
Step 2) prove that, if the formula work for n, it also must work for n+1

Prove that $\sum_{j=1}^{n} j^3 = \frac{n^2(n+1)^2}{4}$

Step 1: Show it works for n=1

$$\sum_{j=1}^{1} j^3 = \frac{(1)^2(1+1)^2}{4}$$

$$1^3 = \frac{(1)(2)^2}{4} = \frac{4}{4} = 1$$

This works.

Step 2: Assume the formula works for n. Show for n+1.

Assuming $\sum_{j=1}^{n} j^3 = \frac{n^2(n+1)^2}{4}$ is up to some value of n. Show it works for the next value of n, i.e. n+1.

$$\sum_{j=1}^{n+1} j^3 = \frac{(n+1)^2(n+1+1)^2}{4}$$

Applied directly.
Using my assumption

$$\sum_{j=1}^{n+1} j^3 = \sum_{j=1}^{n} j^3 + (n+1)^3 = \frac{n^2(n+1)^2}{4} + (n+1)^3$$

I want to show that both expressions give the same result.

$$\frac{n^2(n+1)^2}{4} + (n+1)^3 = (n+1)^2\left[\frac{n^2}{4} + n + 1\right] = \frac{(n+1)^2}{4}[n^2 + 4n + 4] = \frac{(n+1)^2}{4}(n+2)^2$$

$$= \frac{(n+1)^2(n+1+1)^2}{4}$$

What this says is this formula works for n=1, then it must also work for n=2, and if it works for n=2, then must also work for n=3, etc.

Next time, go over exam 3, finish 9.4 (the last section), review for the final.
The final is next Thursday.